Numerical analysis of time discretization of optimal control problems

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Abstract

The three lectures will discuss known results and open problems for the numerical approximation of optimal control problems by a time discretization.

1. **Problems with final state constraints**
   (a) Optimality conditions, shooting function
   (b) Order of approximation when using Runge-Kutta schemes
   (c) Refinement and logarithmic penalty
2. **Problems with running control and state constraints**
   (a) Control constraints
   (b) Mixed control and state constraints
   (c) First-order state constraints
3. **Discussion of some open problems**
   (a) High order state constraints
   (b) Singular arcs
   (c) Integral equations

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Parametric Sensitivity Analysis of NLP Problems and its Applications to Real-Time Controller Design

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Abstract

Nonlinear optimization has grown to a key technology in many areas of application. At the latest since the discretisation of optimal control problems with ODEs, DAEs or PDEs, large and sparse problems are in the focus of interest. For this reason special techniques are implemented in the new nonlinear optimization solver WORHP which is capable of solving problems with more than 1.000.000.000 variables and constraints.

In case of disturbances or perturbations in the system data one often is not able to calculate the optimal solution in advance. Hence in a real process, the possibility of data disturbances force recomputing the problem to preserve stability and optimality, at least approximately.

For this we consider parametric nonlinear programming problems involving a parameter $p \in P \subset \mathbb{R}^{N_p}$. The optimization variable is denoted by $z \in \mathbb{R}^{N_z}$. The *parametric NLP–problem* with equality and inequality constraints is given by

$$(1)\mathbf{NLP(p)} \quad \begin{cases} \min\limits_{z} & F(z,p), \\ \text{subject to} & G_i(z,p) = 0, \quad i = 1,\ldots,N_e, \\ & G_i(z,p) \le 0, \quad i = N_e+1,\ldots,N_c. \end{cases}$$

For a fixed reference or *nominal* parameter $p_0 \in P$, we will study the differential properties of the optimal solutions to the perturbed problems $\mathbf{NLP}(p)$ in a neighbourhood of the nominal parameter p_0. This post optimality analysis leads to an explicit formulae for the so called sensitivity differentials

$$\frac{dz}{dp}(p_0), \tag{2}$$

which can be used for *real-time approximations* of perturbed solutions by applying its first order Taylor expansion:

$$z(p) \approx z_0 + \frac{dz}{dp}(p_0)\,(p - p_0)\,. \tag{3}$$

Based on the general idea (3) several different methods are suggested to calculate higher order approximations of optimal solutions in real-time. The thechniques will be applied to different numerical challanges:
– near optimal feasible solutions of NLP problems in real-time,
– B-step correction step technique in nonlinear programming,
– sub-optimal solution of optimal control problems in real-time,
– on-line trajectory planning,
– near optimal full nonlinear feedback controller,
– adaptive sub-optimal feedback controller design,
– adaptive sub-optimal observer design.

Theoretical backgrounds will be discussed and numerical results will be presented for different test cases.

Theory and Applications of Constrained Optimal Control Problems with Delays

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joint work with *Laurenz Göllmann*

Abstract

This talk addresses optimal control problems with constant delays in state or control variables. The dynamic process is subject to mixed control-state or pure state inequality constraints. Let $x \in \mathbf{R}^\mathbf{n}$ denote the state variable, $u \in \mathbf{R}^\mathbf{m}$ the control variable $d \geq 0$ the state variable delay, and $s \geq 0$ the control variable delay. We consider the following delayed (retarded) control problem:

$$\text{Minimize} \quad J(x,u) = g(x(t_f) + \int_0^{t_f} f_0(x(t), x(t-d), u(t), u(t-s))\, dt$$

subject to

$$\begin{array}{ll} \dot{x}(t) = f(x(t), x(t-d), u(t), u(t-s)), & 0 \leq t \leq t_f\,, \\ x(t) = x_0(t) \quad \forall\ -d \leq t \leq 0, & \varphi(x(t_f)) = 0, \\ u(t) = u_0(t) \quad \forall\ -s \leq t \leq 0, & \\ C(x(t), u(t)) \leq 0\,, \quad S(x(t)) \leq 0\,, & 0 \leq t \leq t_f\,. \end{array}$$

The functions have appropriate dimensions and are assumed to be sufficiently smooth. We sketch the main idea behind numerical methods for solving the delayed control problem. Using suitable discretization schemes the delayed problem is transcribed into a large-scale optimization problem which can be solved by SQP–methods or Interior-Point Methods; cf., the talk by Christof Büskens. We present Pontryagin type Minimum Principles by using the augmentation technique developed in [2, 1]. The basic proof idea consists in transforming the delayed control problem into a standard non-delayed optimal control problem by augmenting the state dimension. The adjoint variable for the delayed problem satisfies an advanced ODE. The multiplier associated with the state inequality constraint is a measure which under certain regularity condition is represented by a regular function; cf. [5, 4].

The main focus in this talk is on applications. Part 1 of the talk treats *mixed control-state constraints*, in particular *pure control constraints*. We discuss the following examples: (1) control (temperature and feeding rate) of continuously stirred chemical reactors, (2) optimal exploitation of renewable resources (fish) and non-renewable resources (oil), and (3) optimal investment and dividend of a firm. Part 2 of the talk focusses on control problems with *pure state constraints*. The following examples are provided: (1) an illustrative academic example, (2) optimal control of the innate immune response, (3) optimal control of a model of climate change with constraints on temperature and CO_2–concentration.

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Numerical schemes for Hamilton-Jacobi equations: control problems and differential games

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Abstract

The course aims to introduce numerical schemes derived by Dynamic Programming for control problems and differential games. We will present the origins of the method and some applications to classical control problems discussing several topics like accuracy, efficiency and convergence. In the last part of the course we will also deal with problems with state constraints and with the approximation of reachable sets.
The outline of the lectures is the following:

Lecture 1
Control problems, Dynamic programming and Hamilton Jacobi equations. Schemes derived via Dynamic Programming. How to build a scheme. General properties: monotonicity, consistency, stability, convergence.

Lecture 2
Approximation of some classical control problems. Some a-priori error estimates. Approximations schemes for pursuit-evasion games. Recent developments and open problems.

Lecture 3
Approximation of reachable sets. Motion planning and minimum time problems. General control problems with state constraints. Feedback synthesis.

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