Pontraygin's Maximum Principle for some families of PDEs

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Outline

1. Motivating example
2. Optimal Control Problem
3. Pontryagin's Maximum Principle for k-symplectic formalism
4. Formalism for OCP governed by implicit partial differential equation

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Rotation of a bipolar rigid molecule

[Boscain, Caponigro, Chambrion, Sigalotti, 2012]

Imagine the following contol PDE:

$$\mathrm{i}\frac{\partial \Psi(t,\theta)}{\partial t} = -\frac{\partial^2 \Psi(t,\theta)}{\partial \theta^2} + u_1(t)\,\Psi(t,\theta)\,\cos\theta + u_2(t)\Psi(t,\theta)\,\sin\theta.$$

where

- u_1 and u_2 are **two control electric fields** taking values in $\mathbb{R}$;
- Ψ is an element in a Hilbert space taking values on $\mathbb{C}$.

Now, add a cost function...for instance:

$$L = \frac{1}{2}\left(u_1^2 + u_2^2\right) .$$

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Classical optimal control problem (OCP)

- A n-dimensional manifold Q.
- The natural tangent bundle projection $\tau_Q \colon TQ \to Q$.
- A **control set** $U \subset \mathbb{R}^l$.
- A **cost function** $G \colon Q \times U \to \mathbb{R}$.

Statement 1 (OCP for (Q, U, X, G, I))

Find $(\gamma, u) \colon I \subset \mathbb{R} \to Q \times U$ joining x_0 and x_f in Q such that

1. it is an integral curve of the vector field X defined along the projection $\pi_1 \colon Q \times U \to Q$, i.e.

$$\dot{\gamma}(t) = X(\gamma(t), u(t));$$

2. it minimizes the functional $\int_I G(\tilde{\gamma}(t), \tilde{u}(t))\mathrm{d}t$ among all the integral curves $(\tilde{\gamma}, \tilde{u})$ of X on $Q \times U$ joining x_0 and x_f.

Crash course on k-symplectic formalism [Awane, 1992]

The k-**tangent bundle** of Q, $T^1_k Q$, is the following Whitney sum:

$$T^1_k Q = TQ \oplus \overset{k}{\cdots} \oplus TQ.$$

The elements of $T^1_k Q$ are k-tuples $(v_{1_q}, \ldots, v_{k_q})$ of vectors in $T_q Q$, $q \in Q$.

The **canonical projection** $\tau^{\mathbf{k}}_{\mathbf{Q}} \colon \mathbf{T^1_k Q} \to \mathbf{Q}$: $\tau^k_Q(v_{1_q}, \ldots, v_{k_q}) = q$.

Local coordinates for $T^1_k Q$: (q^i, v^i_A), $A = 1, \ldots, k$, $i = 1, \ldots, \dim\, Q$.

A k-**vector field** on Q is a section $\mathbf{X} \colon Q \to T^1_k Q$ of τ^k_Q.

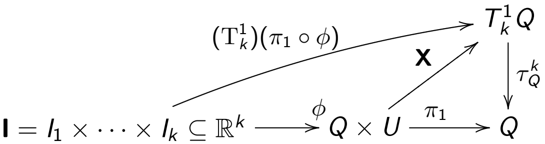
The **canonical projection** $\tau^{\mathbf{k;A}}_{\mathbf{Q}}(\mathbf{v_{1_q}}, \ldots, \mathbf{v_{k_q}}) = \mathbf{v_{A_q}}$ associates $\mathbf{X}$ with

a family of vector fields on Q : $X_A = \tau^{k;A}_Q \circ \mathbf{X} \quad A = 1, \ldots, k$.

An **integral section of X** is a map $\sigma \colon \mathbb{R}^k \to Q$, $\mathbf{t} \to \sigma(\mathbf{t})$ s.t.

$$\mathrm{T}^1_k \sigma = \left(\frac{\partial \sigma}{\partial t^1}, \ldots, \frac{\partial \sigma}{\partial t^k} \right)_{\sigma(\mathbf{t})} = \mathbf{X} \circ \sigma \quad \text{where } \mathbf{t} = (t^1, \ldots, t^k).$$

k-symplectic Optimal Control Problem (k-OCP)



Let $F\colon Q \times U \to \mathbb{R}$ be the **cost function**.

Statement 2 (k-OCP for $(Q, U, \mathbf{X}, F, \mathbf{I})$)

Find a map $\phi = (\sigma, u)\colon \mathbf{I} \to Q \times U$ passing through q_0 and q_f in Q s. t.

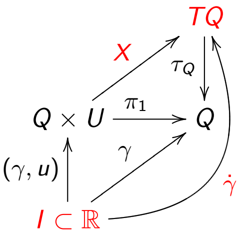
1. it is an integral section of $\mathbf{X} = (X_1, \ldots, X_k)$, i.e.

$$\mathrm{T}^1_k(\pi_1 \circ \phi) = \mathbf{X} \circ \phi, \quad \text{i.e.} \quad \frac{\partial \sigma^i}{\partial t^A}(\mathbf{t}) = X^i_A(\phi(\mathbf{t})) = X^i_A(\sigma(\mathbf{t}), u(\mathbf{t})),$$

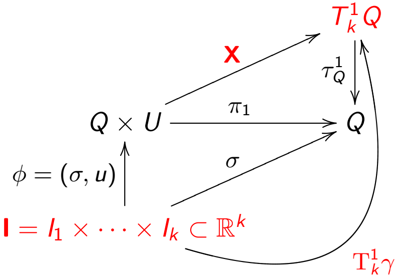
2. it minimizes the functional $\int_{I_1 \times \cdots \times I_k} F(\tilde{\phi}(\mathbf{t}))\mathrm{d}t^1 \wedge \cdots \wedge \mathrm{d}t^k$ among all the integral sections $\tilde{\phi}$ of $\mathbf{X}$ on $Q \times U$ passing through q_0 and q_f.

Comparison between OCP and k-OCP

Classical OCP for ODE



k-symplectic OCP



KEY assumptions to adapt PMP to k-OCP

1. The Lie derivative of the cost function with respect to each X_A is zero, that is, $\mathrm{L}_{\mathbf{X_A}}\mathbf{F} = \mathbf{0}$.

2. The control functions $u\colon \mathbf{I} \to U$ are locally constants.

NECESSARY TRICK: Add k new variables $(q^{0_1}, \dots, q^{0_k})$ s. t. for every $A = 1, \dots, k$

$$\frac{\partial q^{0_A}}{\partial t^A} = F, \quad \frac{\partial q^{0_A}}{\partial t^B} = 0, \qquad \text{for } B \neq A.$$

Note that q^{0_A} for $B \neq A$ satisfies

$$\frac{\partial q^{0_A}}{\partial t^B} = \int_{t_0^A}^{t^A} \left(\frac{\partial F}{\partial q^i} \frac{\partial q^i}{\partial t^B} + \frac{\partial F}{\partial u^a} \frac{\partial u^a}{\partial t^B} \right) (t^1, \dots, s, \dots, t^k) \mathrm{d}s = 0.$$

Note that if both assumptions are satisfied the equations will be immediately satisfied.

Extended k-symplectic OCP, $\widehat{k-OCP}$

- The **extended manifold in k-symplectic formalism**: $\widehat{Q} = \mathbb{R}^k \times Q$.
- The **extended k-vector field $\widehat{\mathbf{X}}$ on** $\widehat{Q}$ is given by $(\widehat{X}_1, \ldots, \widehat{X}_k)$ where

$$\widehat{X}_A = F\delta_A^B \frac{\partial}{\partial q^{0_B}} + X_A = F \frac{\partial}{\partial q^{0_A}} + X_A, \quad \text{for every} \quad A = 1, \ldots, k,$$

where δ_A^B is the Kronecker's delta and F is the cost function.

- First integrate $\dfrac{\partial q^i}{\partial t^A} = X_A^i(q, u)$ and obtain $\phi = (\sigma, u)\colon \mathbf{I} \to Q \times U$.

Then,

$$q^{0_A}(\mathbf{t}) = \int_{t_0^A}^{t^A} F(\sigma(\mathbf{t}), u(\mathbf{t}))\mathrm{d}t^A, \quad \text{for every } A = 1, \ldots, k.$$

Extended k-symplectic OCP, $\widehat{k-OCP}$

Statement 3 ($\widehat{k-OCP}$ for $(\widehat{Q}, U, \widehat{\mathbf{X}}, F, \mathbf{I})$)

Find a map $\widehat{\phi} = (\widehat{\sigma}, u) \colon \mathbf{I} \subset \mathbb{R}^k \to \widehat{Q} \times U$ passing through $(\mathbf{0}, q_0)$ in $\widehat{Q}$ and q_f in Q s.t.

1. it is an integral section of $\widehat{\mathbf{X}} = (\widehat{X}_1, \ldots, \widehat{X}_k)$ defined along the projection $\widehat{\pi}_1 \colon \widehat{Q} \times U \to \widehat{Q}$, i.e

$$\frac{\partial \sigma^{0_B}}{\partial t^A}(\mathbf{t}) = F(\phi(\mathbf{t}))\delta^B_A, \quad \frac{\partial \sigma^i}{\partial t^A}(\mathbf{t}) = X^i_A(\phi(\mathbf{t}));$$

2. it minimizes each functional for $A = 1, \ldots, k$

$$\mathcal{F}_A[\phi](\mathbf{t}) = \int_{t_0^A}^{t^A} F(\phi(t^1, \ldots, \overset{\text{Ath}}{s}, \ldots, t^k))\, \mathrm{d}s,$$

among all the integrals sections $\widehat{\phi}$ of $\widehat{\mathbf{X}}$ on $\widehat{Q} \times U$ passing through q_0 and q_f s.t. $\phi = \pi_{Q\times U} \circ \widehat{\phi}$ for $\pi_{Q\times U} \colon \widehat{Q} \times U \to Q \times U$.

Extended k-symplectic OCP, $k\widehat{-OCP}$

In contrast with classical optimal control theory,

- in k-symplectic formalism the extended optimal control problem and the optimal control problem are **not equivalent**.
- However, solutions to the extended problem $k\widehat{-OCP}$ are also solutions to the original k-symplectic optimal control problem.

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Hamiltonian formulation of $k - \widehat{OCP}$

- Consider k Hamiltonian functions $H_A\colon (T^1_k)^*\widehat{Q} \times U \to \mathbb{R}$:

$$H_A(\widehat{\mathbf{p}}, u) = \langle \widehat{p}^A, \widehat{X}_A(\widehat{q}, u)\rangle = p^A_{0_A} F(q,u) + \sum_{j=1}^{n} p^A_j X^j_A(q,u),$$

- Local coord. $(q^{0_1}, \dots, q^{0_k}, q^1, \dots, q^n, (p^A_{0_1}, \dots, p^A_{0_k}, p^A_1, \dots, p^A_n))$.
- For each control u, the Hamiltonian k-vector field $\widehat{\mathbf{X}}^{*\{u\}}$ must satisfy

$$\mathrm{i}_{\widehat{X}^{*\{u\}}_A}\,\omega^A = \mathrm{d}H^{\{u\}}_A \quad \text{for every } A = 1, \dots, k, \quad (\star)$$

where $\omega^A = (\pi^A)^*\omega$, projection $\pi^A\colon (T^1_k)^*\widehat{Q} \to T^*\widehat{Q}$ onto the Ath-copy.

- Locally $\omega^A = \mathrm{d}q^{0_j} \wedge \mathrm{d}p^A_{0_j} + \mathrm{d}q^i \wedge \mathrm{d}p^A_i$.
- **A solution to $(\star)$ is a solution to Hamilton-De Donder-Weyl eq.**

$$\sum_{A=1}^{k} \mathrm{i}_{\widehat{X}^{*\{u\}}_A}\,\omega^A = \sum_{A=1}^{k} \mathrm{d}H^{\{u\}}_A = \mathrm{d}\left(\sum_{A=1}^{k} H^{\{u\}}_A\right) = \mathrm{d}\mathbf{H}\,.$$

Let us compare ($\star$) and Hamilton-De Donder-Weyl equations

Hamiltonian k-vector field $\widehat{\mathbf{X}}^* = \left(\widehat{X}_1^*, \ldots, \widehat{X}_k^*\right)$ locally expressed as

$$\widehat{X}_A^* = (Y_A)^{0_B} \frac{\partial}{\partial q^{0_B}} + (Y_A)^i \frac{\partial}{\partial q^i} + (Y_A)_{0_B}^C \frac{\partial}{\partial p_{0_B}^C} + (Y_A)_j^C \frac{\partial}{\partial p_j^C}.$$

From ($\star$) we obtain

$$\begin{array}{rclrcl} (Y_A)^{0_B} & = & X_A^{0_B} = F\delta_A^B, & (Y_A)_{0_B}^A & = & 0, \\ (Y_A)^i & = & X_A^i, & (Y_A)_i^A & = & -p_{0_A}^A \dfrac{\partial F}{\partial q^i} - p_j^A \dfrac{\partial X_A^j}{\partial q^i}, \end{array}$$

Note that $(Y_A)_{0_B}^C$, $(Y_A)_j^C$ remain undetermined for $C \neq A$.

Hamilton-De Donder-Weyl equations lead to

$$(Y_A)^{0_B} = X_A^{0_B} = F\delta_A^B, \quad (Y_A)_{0_B}^A = 0, \quad (Y_A)^i = X_A^i,$$
$$\sum_{A=1}^k (Y_A)_i^A = \sum_{A=1}^k \left(-p_{0_A}^A \frac{\partial F}{\partial q^i} - p_j^A \frac{\partial X_A^j}{\partial q^i}\right).$$

Let us make the the Hamiltonian k-vector field $\widehat{\mathbf{X}}^{*\{u\}}$ fully determined

Note that $(T^1_k)^*\widehat{Q} = (T^1_k)^*(\mathbb{R}^k \times Q) \simeq (T^1_k)^*\mathbb{R}^k \times (T^1_k)^*Q$.

$$
\begin{array}{ccccc}
 & & (T^1_k)^*\mathbb{R}^k \times (T^1_k)^*Q & & \\
 & \swarrow^{\mathrm{pr}_1} & & \searrow^{\mathrm{pr}_2} & \\
(T^1_k)^*\mathbb{R}^k & & & & (T^1_k)^*Q \\
\downarrow \pi^{k;C}_{\mathbb{R}^k} & & & & \downarrow \pi^{k;C}_{Q} \\
T^*\mathbb{R}^k & & & & T^*Q
\end{array}
$$

- The conditions

$$\mathrm{T}\left(\pi^{k;C}_{\mathbb{R}^k} \circ \mathrm{pr}_1\right)\left(\widehat{X}^*_A\right) = 0, \quad \mathrm{T}\left(\pi^{k;C}_{Q} \circ \mathrm{pr}_2\right)\left(\widehat{X}^*_A\right) = 0,$$

for every $C \neq A$ **imply locally that** $(Y_A)^C_{0B} = 0$ **and** $(Y_A)^C_j = 0$ **for** $C \neq A$.

k-symplectic PMP

Theorem 3 (k-symplectic Pontryagin's Maximum Principle)

If $\widehat{\phi}^* = (\widehat{\sigma}^*, u^*)\colon I_1 \times \cdots \times I_k \to \widehat{Q} \times U$ is a solution of $\widehat{k - OCP}$ such that F satisfies assumptions 1 and 2, then there exists $(\widehat{\beta}, u)\colon I_1 \times \cdots \times I_k \to (T^1_k)^* \widehat{Q} \times U$ along $\widehat{\sigma}^*$ s.t.

1. $(\pi^A \circ \widehat{\beta}, u)$ *along* $\widehat{\sigma}^*$ *is a solution of* $(\star)$ *for each* $A = 1, \ldots, k$;
2. *the Hamiltonian* $H_A\colon (T^1_k)^* \widehat{Q} \times U \to \mathbb{R}$ *along* $\widehat{\phi}^*$ *is equal to the supremum of* H_A *over the controls a.e. ;*
3. *the supremum of the Hamiltonian* H_A *along* $\widehat{\phi}^*$ *is constant a.e. ;*
4. $\widehat{\beta}^A(\mathbf{t}) \neq 0 \in T^*_{\widehat{\sigma}^*(\mathbf{t})} \widehat{Q}$ *for each* $\mathbf{t} \in I_1 \times \cdots \times I_k$;
5. $\beta^A_{0_1}(\mathbf{t}), \ldots, \beta^A_{0_k}(\mathbf{t})$ *are constant and* $\beta^A_{0_A}$ *is non-positive.*

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4. **Formalism for OCP governed by implicit partial differential equation**

Unified formalism for k-cosymplectic [Rey, Román, Salgado, Vilariño, 2012] to implicit PDEs

- $\mathcal{W} := \mathbb{R}^k \times \left(T^1_k Q \oplus \left(T^1_k\right)^* Q \right)$. Local coordinates (t^B, q^i, v^i_A, p^A_i).
- $\vartheta^A = (\mathrm{pr}_1)^*(\mathrm{d}t^A)$ and $\Omega_A = (\mathrm{pr}_3)^*(\omega_A)$ for $1 \leq A \leq k$, where $\mathrm{pr}_1 \colon \mathcal{W} \to \mathbb{R}^k$, $\mathrm{pr}_3 \colon \mathcal{W} \to \left(T^1_k\right)^* Q$.
- Locally, $\vartheta^A = \mathrm{d}t^A, \quad \Omega_A = \mathrm{d}q^i \wedge \mathrm{d}p^A_i$.
- The **coupling function** $\mathcal{C}$ on $\mathbb{R}^k \times \left(T^1_k Q \oplus \left(T^1_k\right)^* Q \right)$:

$$\mathcal{C}(\mathbf{t}, \mathbf{v}_q, \mathbf{p}_q) = \sum_{A=1}^{k} p^A_q(v_{A_q}) = \sum_{A=1}^{k} (p^A_i v^i_A).$$

- Given a **Lagrangian function** $\mathbb{L}$ on $\mathbb{R}^k \times T^1_k Q$, the **Hamiltonian function H** on $\mathbb{R}^k \times \left(T^1_k Q \oplus \left(T^1_k\right)^* Q \right)$ is defined as follows

$$H = \mathcal{C} - (\mathrm{pr}_1 \times \mathrm{pr}_2)^* \mathbb{L}.$$

Locally, $H(t, q^i, v^i_A, p^A_i) = p^A_i v^i_A - \mathbb{L}(t, q^i, v^i_A)$.

Unified formalism for k-cosymplectic to implicit PDEs

Dynamical problem in the Skinner-Rusk formalism for k-cosymplectic field theories

Find integral sections $\phi\colon \mathbb{R}^k \to \mathcal{W}$ of an integrable k-vector field $\mathbf{Z} = (Z_1, \ldots, Z_k)$ on $\mathcal{W}$ s.t.

$$\sum_{A=1}^{k} \mathrm{i}_{Z_A}\Omega_A = \mathrm{d}H - \sum_{A=1}^{k} \frac{\partial H}{\partial t^A}\vartheta^A, \quad \mathrm{i}_{Z_A}\vartheta^B = \delta_A^B.$$

Unified formalism for k-cosymplectic to implicit PDEs

Definition 4

An **implicit dynamical system** $(\mathbb{L}, M)$ is described by the submanifold

$$M = \{(t^B, q^i, v_A^i) \in \mathbb{R}^k \times (T_k^1)Q \mid \Psi^\alpha(t^B, q^i, v_A^i) = 0,\ 1 \leq \alpha \leq s\},$$

where $\mathrm{d}\Psi^1 \wedge \cdots \wedge \mathrm{d}\Psi^s \neq 0$, and a Lagrangian function $\mathbb{L} \in \mathcal{C}^\infty(M)$.

Using the natural embedding $\iota^M \colon M \hookrightarrow \mathbb{R}^k \times T_k^1 Q$, on the k-symplectic implicit bundle $\mathcal{W}^M = M \times_Q (T_k^1)^* Q$ we have:

$$\mathcal{C}^{\mathcal{W}^M} = (\mathrm{i}^M)^*(\mathcal{C}), \quad \vartheta^A_{\mathcal{W}^M} = (\mathrm{i}^M)^*(\vartheta^A), \quad \Omega_A^{\mathcal{W}^M} = (\mathrm{i}^M)^*(\Omega_A).$$

Problem of describing the dynamics of $(\mathbb{L}, M)$

$$\sum_{A=1}^k \mathrm{i}_{Z_A}\Omega_A^{\mathcal{W}^M} = \mathrm{d}H_{\mathcal{W}^M} - \sum_{A=1}^k \frac{\partial H_{\mathcal{W}^M}}{\partial t^A}\vartheta^A_{\mathcal{W}^M}, \quad \mathrm{i}_{Z_A}\mathrm{d}t^B = \delta_A^B.$$

Unified formalism for OCP governed by an implicit PDE

➢ Let C be the control bundle with natural coordinates (t^A, q^i, u^a).

$$M_C = \{(t^B, u^a, q^i, v^i_A) \in C \times_Q T^1_k Q \mid \Psi^\alpha(t^B, u^a, q^i, v^i_A) = 0,\ 1 \leq \alpha \leq s\}.$$

➢ **Implicit optimal control problem**: $(\mathbb{L}, M_C)$, where $\mathbb{L} \in \mathcal{C}^\infty(M_C)$.

➢ The dynamics of $(\mathbb{L}, M_C)$: for a k-vector field $\mathbf{Z}$ on $\mathcal{W}^{M_C} = M_C \times_Q (T^1_k)^* Q$:

$$\sum_{A=1}^{k} \mathrm{i}_{Z_A}\left(\Omega_A^{\mathcal{W}^{M_C}}\right) = 0, \quad \mathrm{i}_{Z_A}\vartheta^B_{\mathcal{W}^{M_C}} = \delta^B_A,$$

➢ Equivalently, there exists a k-vector field $\mathbf{Z}$ on $C \times_{\mathbb{R}^k \times Q} \mathcal{W}$ such that

(i) $\mathbf{Z}$ is tangent to $\mathcal{W}^{M_C}$;

(ii) the 1-form $\sum_{A=1}^{k} \mathrm{i}_{(Z_A)}(\sigma^*_{\mathcal{W}}(\Omega_A))$ is null on the k-vector fields tangent to $\mathcal{W}^{M_C}$.

Unified formalism for OCP governed by an implicit PDE

- There exist $\lambda_\alpha \in \mathcal{C}^\infty(C \times_{\mathbb{R}^k \times Q} \mathcal{W})$, to be determined, s. t. on $\mathcal{W}^{M_C}$

$$\sum_{A=1}^{k} \mathrm{i}_{(Z_A)}(\sigma^*_{\mathcal{W}}(\Omega_A)) = \mathrm{d}H_{\mathcal{W}^{M_C}} - \frac{\partial H_{\mathcal{W}^{M_C}}}{\partial t^A}\vartheta^A + \lambda_\alpha \mathrm{d}\Psi^\alpha - \lambda_\alpha \frac{\partial \Psi^\alpha}{\partial t^A}\vartheta^A$$

- In coordinates $(t^B, u^a, q^i, v^i_A, p^A_i)$ in $C \times_{\mathbb{R}^k \times Q} \mathcal{W}$, find k vector fields

$$Z_A = (Z_A)^B_t \frac{\partial}{\partial t^B} + (Z_A)^a \frac{\partial}{\partial u^a} + (Z_A)^i \frac{\partial}{\partial q^i} + (Z_A)^i_B \frac{\partial}{\partial v^i_B} + (Y_A)^B_i \frac{\partial}{\partial p^B_i} \; ,$$

verifying the equation

$$\sum_{A=1}^{k} (Y_A)^A_i = \frac{\partial \mathbb{L}}{\partial q^i} - \lambda_\alpha \frac{\partial \Psi^\alpha}{\partial q^i}, \qquad (Z_A)^i = v^i_A,$$

$$p^A_i = \frac{\partial \mathbb{L}}{\partial v^i_A} - \lambda_\alpha \frac{\partial \Psi^\alpha}{\partial v^i_A}, \qquad 0 = \frac{\partial \mathbb{L}}{\partial u^a} - \lambda_\alpha \frac{\partial \Psi^\alpha}{\partial u^a}.$$

together with the tangency conditions on $\mathcal{W}^{M_C}$

$$0 = Z_A(\Psi^\alpha) = (Z_A)^A_t \frac{\partial \Psi^\alpha}{\partial t^A} + (Z_A)^i \frac{\partial \Psi^\alpha}{\partial q^i} + (Z_A)^a \frac{\partial \Psi^\alpha}{\partial u^a} + (Z_A)^i_B \frac{\partial \Psi^\alpha}{\partial v^i_B}$$

Back to the example

$$\mathrm{i}\frac{\partial\Psi(t,\theta)}{\partial t} = \left(-\frac{\partial^2\Psi(t,\theta)}{\partial\theta^2} + u_1(t)\cos\theta\,\Psi(t,\theta) + u_2(t)\sin\theta\,\Psi(t,\theta)\right),$$

- **2-symplectic formalism where $t^1 = t$ and $t^2 = \theta$.**
- Rename variables: $q^1 = \mathrm{Re}\Psi, \quad q^2 = \mathrm{Im}\Psi$.
- Consider a 6-dimensional manifold Q with local coordinates

$$\left(q^1, q^2, q^3 = \frac{\partial q^1}{\partial t}, q^4 = \frac{\partial q^2}{\partial t}, q^5 = \frac{\partial q^1}{\partial\theta}, q^6 = \frac{\partial q^2}{\partial\theta}\right).$$

- The local coordinates for $C \times_Q T^1_2 Q$ are $(t^1, t^2, u^1, u^2, q^i, v^i_1, v^i_2)$.
- The submanifold M_C of $C \times_Q T^1_2 Q$ is implicitly defined by

$$\begin{array}{llll}
\Psi^1 = v^1_1 - q^3, & \Psi^4 = v^2_2 - q^6, & \Psi^7 = -q^3 - v^6_2 + u_1 q^2\cos\theta + u_2 q^2\sin\theta\,, \\
\Psi^2 = v^2_1 - q^4, & \Psi^5 = v^3_2 - v^5_1, & \Psi^8 = q^4 - v^5_2 + u_1 q^1\cos\theta + u_2 q^1\sin\theta\,. \\
\Psi^3 = v^1_2 - q^5, & \Psi^6 = v^4_2 - v^6_1, &
\end{array}$$

Back to the example

Thus,

$$
\begin{aligned}
Z_A = {} & \frac{\partial}{\partial t^A} + v^i_A \frac{\partial}{\partial q^i} + (D_A)_a \frac{\partial}{\partial u_a} + v^3_A \frac{\partial}{\partial v^1_1} + v^4_A \frac{\partial}{\partial v^2_1} + v^5_A \frac{\partial}{\partial v^1_2} + v^6_A \frac{\partial}{\partial v^2_2} \\
& + (F_A)^5_1 \left(\frac{\partial}{\partial v^3_2} + \frac{\partial}{\partial v^5_1} \right) + (F_A)^6_1 \left(\frac{\partial}{\partial v^4_2} + \frac{\partial}{\partial v^6_1} \right) \\
& + \Big(v^4_A + (D_A)_1 q^1 \cos\theta - \delta^A_2 u_1 q^1 \sin\theta + v^1_A u_1 \cos\theta + (D_A)_2 q^1 \sin\theta + \delta^A_2 u_2 q^1 \cos\theta \\
& + v^1_A u_2 \sin\theta \Big) \frac{\partial}{\partial v^5_2} + \Big(-v^3_A + (D_A)_1 q^2 \cos\theta - \delta^A_2 u_1 q^2 \sin\theta + v^2_A u_1 \cos\theta + \\
& + (D_A)_2 q^2 \sin\theta + \delta^A_2 u_2 q^2 \cos\theta + v^2_A u_2 \sin\theta \Big) \frac{\partial}{\partial v^6_2} \\
& + (F_A)^3_1 \frac{\partial}{\partial v^3_1} + (F_A)^4_1 \frac{\partial}{\partial v^4_1} + (G_A)^A_i \frac{\partial}{\partial p^A_i},
\end{aligned}
$$

where $t^1 = t$, $t^2 = \theta$, $(D_1)_1 = \cos\theta$, $(D_1)_2 = \sin\theta$ and $(D_2)_1 \sin\theta - (D_2)_2 \cos\theta = -p^2_6 q^2 - p^2_5 q^1$, plus some other equations.

Future research lines

- To find the way to successfully extend these results to any control system regardless of the nature of the cost function. The main difficulty is to obtain a compatible system of partial differential equations after extending the original control system.

- To apply the constraint algorithm for k-presymplectic Hamiltonian systems will characterize the extremals of optimal control problems governed by PDEs.

Thank you!!!!