First Order Necessary Conditions for Optimal Control Problems with State, Control, and Mixed Constraints

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Optimal Control Problem

state $x \in \mathbb{R}^n$, control $u \in \mathbb{R}^m$

Dynamics and Boundary conditions

$$\dot{x}(t) = f(t, x(t), u(t)), \quad \text{a.e. } t \in [0,1] \quad (f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n)$$

$$(x(0), x(1)) \in E \quad (E \subset \mathbb{R}^n \times \mathbb{R}^n)$$

Control and State constraints

$$u \in \mathcal{U}(t) \quad \text{a.e. } t \in [0,1] \quad (\mathcal{U}(t) \subset \mathbb{R}^m)$$

$$b(t, x(t), u(t)) \in \mathcal{B}(t) \quad (b_1 \leq 0, b_2 = 0, \mathcal{B}(t) = (-\infty, 0] \times \{0\})$$

$$h(t, x(t)) \leq 0 \quad (h : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R})$$

Minimize

$$g(x(0), x(1))$$

Optimal Control Problem

state $x \in \mathbb{R}^n$, control $u \in \mathbb{R}^m$ (I will assume all the data is smooth)

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What kind of Necessary Conditions should we expect?

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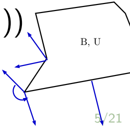
$$\partial_x(\mathcal{I}_{(-\infty,0]}h(t,*) + \mathcal{I}_{\mathcal{B}(t)}(b(t,*)) = \chi \cdot h_x(t,*) + \nu \cdot b_x(t,*)$$

where $\chi \in N_{(-\infty,0]}(h(t,*)) \Rightarrow \chi \geq 0$ and $\nu \in N_{\mathcal{B}(t)}(b(t,*))$



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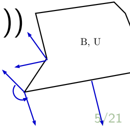

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- $H(t,x,q,\nu,u) = q(t) \cdot f(t,x,u) - \nu \cdot b(t,x,u)$
- $-\dot{p}(t) = H_x(t,x_*,q,\nu,u_*)$ and ...

PMP (Multipliers)

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Then (under appropriete assumptions (later)) there exist:

Adjoint (costate) function $p(.) \in W^{1,1}([0,1],\mathbb{R}^n)$

Multipliers associeted to the constraints

$$\nu \in L^1(0,1) : \nu \in N_{\mathcal{B}(t)}(b(x_*,u_*)) \Rightarrow \nu(t) = 0 \text{ if } b(x_*,u_*) \in \mathring{\mathcal{B}}(t))$$
$$\mu \in \mathcal{C}^+(0,1) : \text{ supp } \mu \subseteq \{t : h(t,x_*(t)) = 0\}$$

Boundary conditions (Normality) $\lambda_0 = \{0,1\}$

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$$\text{Define: } q(t) := \begin{cases} p(t) + \int_{[0,t)} \nabla_x h(x_*)\ d\mu & t \in [0,1) \\ p(t) + \int_{[0,1]} \nabla_x h(x_*)\ d\mu & t = 1 \end{cases}$$

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Maximum Principle

$$\max_{u \in \mathcal{U}(t)} H([t], u) = H([t], u_*(t))$$

Outline

Regularity Conditions

Classical hypotheses under which Necessary Conditions are derived include: [1]

Maximum Rank Condition (Hestenes, 1966)

$\exists\ c > 0$:

$$A = b_u(t, x, u) \implies \det A^T A \geq c$$

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[1] It is sufficient to check the conditions for (t, x, u) near the optimal solution, and for which constraints are active.

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Schwarzkopf Conditions

Necessary conditions for OCPs with mixed constraints can be derived under the following hypotheses:

Covering Hypothesis and Convexity

1. $b(t, x_*(t), u_*(t)) + \delta\mathbb{B} \subseteq b(t, x_*(t), \mathcal{U}(t))$
2. $\{(f(t, x, u), b(t, x, u)) : u \in \mathcal{U}(t)\}$ is convex

Schwarzkopf, "Relaxed control problems with state equality constraints", 1975.

Vinter, De Pinho, "A maximum principle for OCP with mixed constraints", 2001.

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Such approach leads to a very complicated analysis and does not seem to give any relation between Schwarzkopf condition and standard conditions. Moreover

$$b(t,x,u) = a_0(t,x) + a_1(t,x)u \;\Rightarrow\; u_*(t) \in \operatorname{int} \mathcal{U}(t)$$

No reference to $\mathcal{B}(t)$; Unusual convexity assumption (Necessary?).

A Linear mixed constraint

Assume $u \in U$ compact and $\mathcal{B}(t)$ closed, convex set. Then

Covering Hypothesis

- $b(t, x_*(t), u_*(t)) + \delta\mathbb{B} \subseteq b(t, x_*(t), \mathcal{U}(t))$
- $b(t, x, u) = a_0(t, x) + a_1(t, x)u$

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Constraint qualification

$\forall\ \nu \in N_{\mathcal{B}(t)}(b(t, x, u))$ and $\gamma \in N_U(u)\ \exists\ k(.) \in L^1$:

$$\|\nu\| \leq k(t)\|\nu \cdot b_u(t, x, u) + \gamma\|$$

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Differential Inclusions

$$\min\left\{\begin{array}{l} g(x(0,x(1)) : \\ \dot{x} = f(t,x,u),\;\; u \in \mathcal{U}(t) \text{ and } b(t,x,u) \in \mathcal{B}(t), \\ h(t,x) \leq 0,\;\; (x(0),x(1)) \in E. \end{array}\right\}$$

Mixed and control constraints can be absorbed in the dynamic:

$$F(t,x) = \{f(t,x,u) : u \in \mathcal{U}(t), b(t,x,u) \in \mathcal{B}(t)\}$$

Bettiol, Boccia, Vinter, "Stratified Necessary Conditions for Differential Inclusions with State Constraints", 2013.

Regularity conditions for $b(.)$, such as the Mangasarian-Fromowitz condition, can be translated into regularity conditions for $F(.)$.

Differential Inclusions

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Precisely, we require $F(.)$ to be Lipschitz, or, more in general, pseudo-Lipschitz

$$F(t,x) \cap R(t) \subset F(t,x') + k(t)|x - x'|$$

where $k(.) \in L^1$. What is $R(.)$?
$R(t) = \mathbb{R}^n, \epsilon\mathbb{B}, \uparrow \mathbb{R}^n$, as long as $F(t,x) \cap R(t) \neq \emptyset$.

Differential Inclusions

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Assume that the usual constraint qualification holds, namely

$\forall\ \nu \in N_{\mathcal{B}(t)}(b(t, x, u))$ and $\gamma \in N_U(u)\ \exists\ k(.) \in L^1$:

$$\|\nu\| \leq k(t) \|\nu \cdot b_u(t, x, u) + \gamma\|.$$

Then,

$$(\alpha, \beta) \in N_{GrF(t,.)}(x, \dot{x}) \Rightarrow \|\alpha\| \leq k(t) \|\beta\|$$

$$\Leftrightarrow$$

$$F(t, x) \cap R(t) \subset F(t, x') + k(t)|x - x'|$$

Necessary Conditions

Let x_* be a minimum for

$$\min \left\{ \begin{array}{l} g(x(0,x(1)) : \quad \dot{x}(t) = f(t,x,u), \ u \in \mathcal{U}(t) \\ b(t,x,u) \in \mathcal{B}(t), \ h(t,x) \leq 0, \ (x(0),x(1)) \in E. \end{array} \right\}$$

$\exists$ a multiplier $(p, \mu, \lambda_0) \neq (0,0,0)$:

- $(p(0), -q(1)) \in \lambda_0 \nabla g(x_*(0), x_*(1)) + N_E(x_*(0), x_*(1))$
- $-\dot{p}(t) = \nabla_x H([t], u_*(t))$
- $\max_{u \in \mathcal{U}(t)} H([t], u) = H([t], u_*(t))$

$$\text{Define: } q(t) := \begin{cases} p(t) + \int_{[0,t)} \nabla_x h(x_*) \, d\mu & t \in [0,1) \\ p(t) + \int_{[0,1]} \nabla_x h(x_*) \, d\mu & t = 1 \end{cases}$$

$$H([t], u) := q(t) \cdot f(x_*, u) - \nu(t) \cdot b(x_*, u)$$

Is there any class of problems for which the measure μ is "nice"?

Say, μ absolutely continuous w.r.t. the Lebesgue measure:

$$q(t) = p(t) + \int_{[0,t)} \nabla_x h(\tau, x_*) \, \xi(\tau) d\tau$$

For some L^1 function $\xi(.)$.

SEIR Model

- Neilan, Lenhart, "An Introduction to Optimal Control with an Application in Disease Modeling"
- Kornienko, MdR de Pinho, "On first order state constrained optimal control problems"

$$
\begin{cases}
\text{Minimize } \int_0^T (AI(t) + u^2(t))dt \\
\text{subject to} \\
\qquad \dot{S}(t) = bN(t) - dS(t) - cS(t)I(t) - u(t)S(t), \\
\qquad \dot{E}(t) = cS(t)I(t) - (e+d)E(t), \\
\qquad \dot{I}(t) = eE(t) - (g+a+d)I(t), \\
\qquad \dot{N}(t) = (b-d)N(t) - aI(t), \\
\qquad S(t) \leq S_{max}, \\
\qquad S(t)u(t) \leq V_0 \\
\qquad u \in [0,1] \\
\qquad (S(0), E(0), I(0), N(0)) = (S_0, E_0, I_0, N_0)
\end{cases}
$$

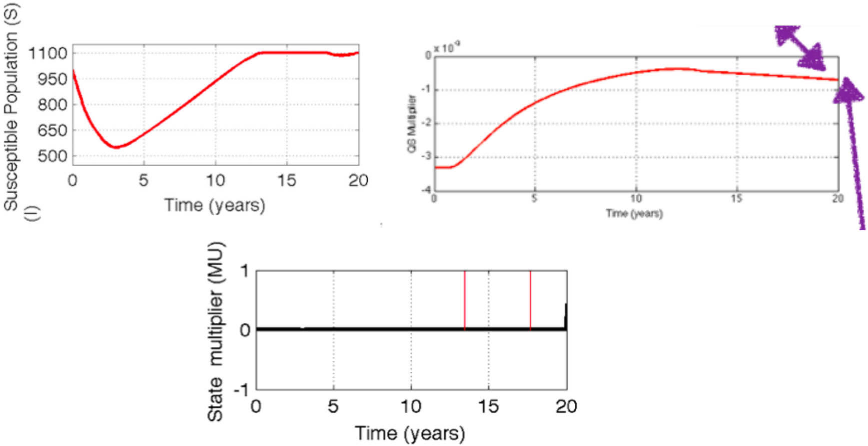
$$x = (S, E, I, N) \ \Rightarrow \ \dot{x}(t) = f_0(x) + f_1(x)u$$

Necessary Conditions

$$\begin{cases} \text{Minimize } \displaystyle\int_0^T (AI(t) + u^2(t))dt \\ \text{subject to} \\ \qquad \dot{x}(t) = f_0(x) + f_1(x)u \\ \qquad u \in [0,1] \\ \qquad h(x) = (1,0,0,0)\cdot(S,E,I,R) \leq S_0 \\ \qquad b(t,x,u) = (1,0,0,0)\cdot(S,E,I,R)u \leq V_0 \\ \qquad x(0) = x_0 \end{cases}$$

- $\lambda = 1$;
- $q(t) = p(t) + \int_{[0,t)} \nabla h(x_*)\, d\mu$;
- $q(T) = 0$;
- $u_*(t) = \max\left\{0, \min\{1, \frac{-q_S(t)S_*(t)}{2}\}\right\}$

An Example (From I. Kornienko, MdR de Pinho, MTNS 2014)



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- We extended current available Theory on necessary conditions by allowing a combination of mixed constraints and pure state constraints.
- We showed that mixed constraint problems can always be reformulated as differential inclusion (mixed-constraint-free) problem. Open question: when state constraints are present, can we identify a class of problems for which NC are "nice"?
- We showed that some relations exist between classical hypotheses on the mixed constraint. Open question:

$$u \mapsto b(t, x, u) \text{ linear (convex)} \quad \rightarrow \quad b(t, x, U) \text{ convex}$$

Thank you

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