L^1-minimization in space mechanics: Old and new

AIMS
Madrid, July 2014

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L^1-minimization

Sparsity of solutions. For fixed final time t_f, consider

$$\ddot{q}(t) = -\nabla V(q(t)) + u(t), \quad |u(t)| \leq 1,$$

$$\int_0^{t_f} |u(t)|\,\mathrm{d}t \to \min,$$

where q and u are valued in $\mathbf{R}^m$ and

$$|u| := |u|_2 = \sqrt{u_1^2 + \cdots + u_m^2}.$$

Pontrjagin maximum principle indicates the possibility of zero control ("coast") arcs as

$$H(q, v = \dot{q}, u, p_q, p_v) = p^0|u| + p_q v + p_v(-\nabla V(q) + u) \leq p_q v - p_v \nabla V(q) + |u|(|p_v| + p^0).$$

Singular arcs. In constrat with finite dimensional optimization, there may also exist singular arcs along which $|u| \in (0, 1)$.

L^1-minimization

Variable mass mechanical systems. Consider

$$\ddot{q}(t) = -\nabla V(q(t)) + \frac{u(t)}{M(t)}, \quad |u(t)| \le 1,$$

$$\dot{M}(t) = -|u(t)|.$$

Minimization of consumption. Given boundary conditions and fixed t_f, equivalence of Lagrange

$$\|u\|_1 := \int_0^{t_f} \sqrt{u_1^2(t) + \cdots + u_m^2(t)}\,\mathrm{d}t \to \min$$

and Mayer optimal control problems:

$$M(t_f) \to \max\,.$$

L^1-minimization

Controllability properties. With $x := (q,\dot q) \in \mathbf{R}^n$, $n = 2m$,

$$\dot x(t) = 1 \cdot F_0(x(t)) + \frac{1}{M(t)} \sum_{i=1}^{m} u_i(t) F_i(x(t)),$$

$$\dot M(t) = -|u(t)|$$

where

$$F_0(q,\dot q) := \dot q \frac{\partial}{\partial q} - \nabla V(q) \frac{\partial}{\partial \dot q}, \quad F_i(q,\dot q) := \frac{\partial}{\partial \dot q_i}, \quad i = 1,\dots,m.$$

Lemma. The Lie algebra generated by F_0, $F_1,\dots,F_m$ is everywhere of maximal rank.

$\Longrightarrow$ Controllability provided additional assumptions on the drift F_0.

Circular restricted three body problem

Continuous vs. impulsive thrust. American and Russian studies of low thrust missions (as opposed to chemical boosts) since the 60's.

Controlled 2/3 BP. For mass ratio $\mu \in [0,1]$, consider

$$\ddot{q}(t) = -\nabla_q V_\mu(t, q(t)) + \frac{\varepsilon u(t)}{M(t)}, \quad |u(t)| \le 1,$$

$$\dot{M}(t) = -|u(t)|$$

where ($q \in \mathbf{R}^2 \simeq \mathbf{C}$)

$$V_\mu(t,q) := -\frac{1-\mu}{|q+\mu e^{it}|} - \frac{\mu}{|q-(1-\mu)e^{it}|}.$$

Remark. 2BP controlled problem for $\mu = 0$ (or 1): $V_0(t,q) =: V(q)$.

$\Longrightarrow$ Min. consumption: L^1-minimization.

Circular restricted three body problem

Transfer between periodic orbits, low thrust.

– Deep Space 1 (NASA, 1998-2001)

– SMART1 (ESA, 2003-2006)

– Hayabusa (JAXA, 2003-2010)

– Dawn (NASA, 2007-2015)

– GOCE (ESA, 2009-2013)

– LISA Pathfinder (ESA & NASA, 2015-)

– BepiColombo (ESA & JAXA, 2016-)

...

→ Project with CNES (4-body model, averaging), 2013-2016.

Old (and less old) references

[1] Robbins, H. M. Optimality of intermediate-thrust arcs of rocket trajectories. *AIAA J.* **6** (1965), no. 3, 1094–1098.

"Lawden's spiral (...) is non optimal. Although optimal intermediate-thrust arcs exist, they seem to be without practical significance because of the restrictive junction conditions."

[2] Marchal, C. Chattering arcs and chattering controls. *J. Optim. Theory Appl.* **15** (1975), no. 5, 633–666.

[3] Zelikin, M. I.; Borisov, V. *Theory of chattering control.* Birkhäuser, 1994.

[4] Gergaud, J.; Haberkorn, T. Homotopy Method for minimum consumption orbit transfer problem. *ESAIM Control Optim. and Calc. Var.* **12** (2006), no. 2, 294–310.

Singularities of the characteristics

Pontrjagin maximum principle. If u is an L^1-optimal control, $\exists$ Lipschitz $(p, p_M) : [0, t_f] \to (\mathbf{R}^5)^*$ such that a.e.

$$\dot{x}(t) = \frac{\partial H}{\partial p}(x(t), M(t), u(t), p(t), p_M(t)), \quad \dot{M}(t) = \frac{\partial H}{\partial p_M}(x(t), M(t), u(t), p(t), p_M(t)),$$

$$\dot{p}(t) = -\frac{\partial H}{\partial x}(x(t), M(t), u(t), p(t), p_M(t)), \quad \dot{p_M}(t) = -\frac{\partial H}{\partial M}(x(t), M(t), u(t), p(t), p_M(t))$$

and

$$H(x(t), M(t), u(t), p(t), p_M(t)) = \max_{|v| \leq 1} H(x(t), M(t), v, p(t), p_M(t))$$

where $x = (q, v)$ and

$$H(x, M, u, p, p_M) := p_q v + p_v(-\nabla V(q) + \frac{u}{M}) - p_M |u|.$$

Singularities of the characteristics

Reduction to a single-input system. Set $\rho := |u|$,

$$H = p_q v - p_v(\nabla V(q) + \frac{u}{M}) - p_M|u| \leq H_0 + \rho H_1$$

with

$$H_0(x, M, p, p_M) := p_q v - p_v \nabla V(q), \quad H_1(x, M, p, p_M) := \frac{\sqrt{p_{v_1}^2 + p_{v_2}^2}}{M} - p_M.$$

Along the optimum,

$$H = \max_{\rho \in [0,1]} H_0 + \rho H_1 = H_0 + (H_1)_+$$

$\Longrightarrow$ Two singularities: $p_v = (0,0)$ and $H_1 = 0$ (codimension 2 and 1, *resp.*)

$\Longrightarrow$ Information encoded by the Poisson brackets of H_0 and H_1 (not a vector field lift).

Singularities of the characteristics

π-singularities.

Lemma. For a fixed final time t_f larger than the minimum time, there are no abnormal extremals.

Prop. Zeros of p_v are isolated and correspond to a discontinuity in the control angle (jump of angle π: $u \to -u$).

Sketch of proof. Normality + maximum rank of $\{F_1, F_2, [F_0, F_1], [F_0, F_2]\}$.

Cor. Functions p_v and H_1 vanish simultaneously at most at one instant. Accordingly, (i) $\rho = |u|$ is continuous and equal to 1 through any π-singularity, (ii) no π-singularity along singular arcs.

Sketch of proof. $\dot{H}_1(\bar{t}\pm) = \pm\frac{|\dot{p}_v|}{M}(\bar{t})$, $\dot{p}_v(\bar{t}) \neq 0$ (touch point).

Singularities of the characteristics

Regular switches.

Lemma. At points s.t. $H_{01} \neq 0$, ρ switches from 0 to 1 (or conversely).

Sketch of proof. $\dot{H}_1 = \{H_0 + \rho H_1, H_1\} = \{H_0, H_1\}$.

Singular arcs.

Prop. (Robbins'1965) Singular arcs are of order at least two. In particular, order two arcs are defined whenever $p_v q \neq 0$.

Sketch of proof. Along a singular arc, $H_1 = H_{01} = 0$. Moreover,

$$H_{101} = \frac{H_{01}}{M}, \quad H_{1001} = \frac{H_{01}}{M^2},$$

H_{10001} is equal to $p_v q$ up to some positive constant, and

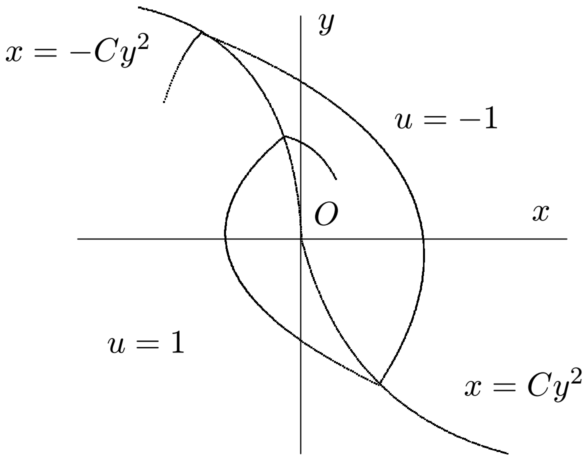
$$H_{00001} + \rho H_{10001} = 0.$$

Singularities of the characteristics

Fuller.

Lemma. No (non-saturating) junction of bang and singular arcs is possible.

$\Longrightarrow$ Fuller phenomenon ("chattering"): Accumulation of switchings points.



Fuller, A. T. Absolute optimality of nonlinear control systems with integral-square error criterion. *J. Electr. Control* **17** (1964), 301-317.

Second order conditions

Fields of extremals. (i) Consider (for fixed t_f)

$$\dot{x}(t) = f(x(t), u(t)), \quad x(t) \in X, \quad u(t) \in U,$$

$$\int_0^{t_f} f^0(x(t), u(t))\,\mathrm{d}t \to \min$$

and assume that the maximized normal Hamiltonian is well defined and smooth:

$$h(x,p) := \max_{u \in U} H(x,u,p) = \max_{u \in U} -f^0(x,u) + pf(x,u).$$

(ii) Let $\mathscr{L} \subset \mathbf{R} \times T^*X$ be a submanifold on which $h\mathrm{d}t - p\mathrm{d}x$ is exact.

Theorem. Assume that, for each $t \in [0, t_f]$, $\Pi : \mathscr{L}_t \to X$, $(x,p) \mapsto x$, is a diffeo.; then, any trajectory of $\overrightarrow{h}$ on T^*X projects onto a trajectory in X optimal w.r.t. all admissible trajectories with same endpoints.

Remark. Works locally by restricting X to some open nbd of a given trajectory ($\mathscr{C}^0$-local optimality).

Second order conditions

Bang arcs. Excluding π-singularities, the absence of conjugate point along the whole arc is sufficient to devise locally a field of extremals (Sarychev'1982). In terms of Jacobi field $\delta z = (\delta x, \delta p)$,

$$\delta \dot{z}(t) = \overrightarrow{h}'(z(t))\delta z(t), \quad \delta x(0) = \delta x(t_c) = 0.$$

Singular arcs. Similar test, well known for order 1 singular arcs (Bonnard-Kupka'1993): Consider

$$h_s(x,p) := H(x, u_s(x,p), p), \quad u_s(x,p) := -\frac{H_{001}}{H_{101}}(x,p)$$

with $H_{101} > 0$ ("hyperbolic case") and appropriate additional constraints for the linearized system. Second order singular: See Dixon & Breakwell (1971).

Remark. Generalized Legendre condition:

$$H_{10001} = (-1)^q \frac{\partial}{\partial \rho} \frac{\mathrm{d}^{2q}}{\mathrm{d}t^{2q}} \frac{\partial H}{\partial \rho}\bigg|_{q=2} < 0, \quad \rho_s := -\frac{H_{00001}}{H_{10001}}.$$

Second order conditions

Bang-bang arcs. Schättler's approach for broken extremals: Include regular switchings t_i by

(i) conjugate point test on each (t_i, t_{i+1})

(ii) transversality condition on switchings in (t, x)-space

Update formula for Jacobi fields includes a jump at regular switchings:

$$\delta z(t_i+) = (I + \Delta_i)\delta z(t_i-), \quad \Delta_i = \frac{\overrightarrow{H_1} H_1'}{H_{01}}(z(t_i)).$$

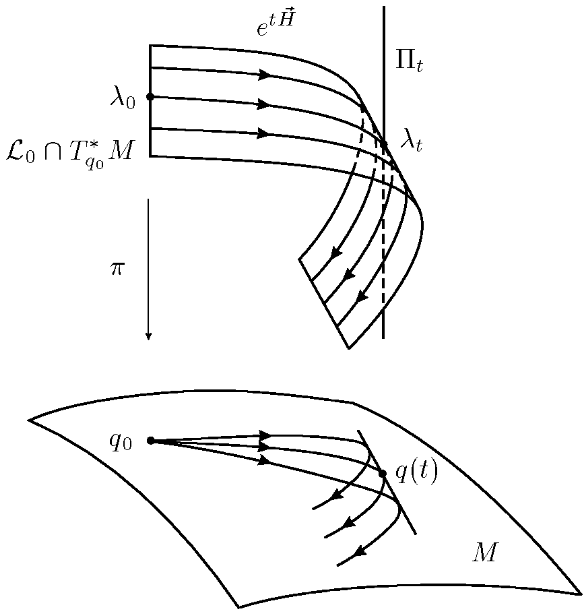
$$(H_1 = (H_0 + H_1) - H_0)$$

Remark. Optimality check not reducible to a finite dimensional problem: Conjugate time at or between switching times.

Second order conditions

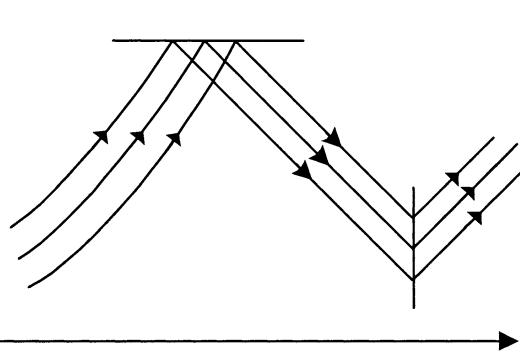
Conjugacy as fold singularity.

Smooth fold



Agrachev, A. A.; Sachkov, Y. L. (2004)

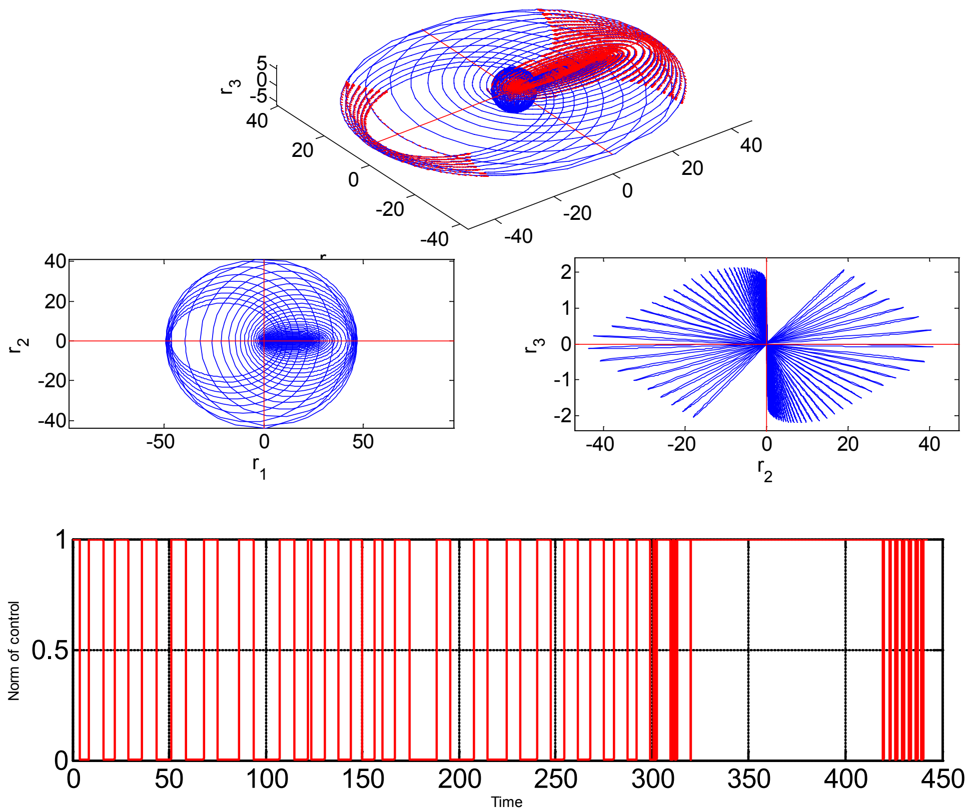
Broken fold



Schättler, H.; Noble, J. (2002)

Second order conditions

Bang-bang arcs. Numerical results (2BP orbit transfer, 3D).



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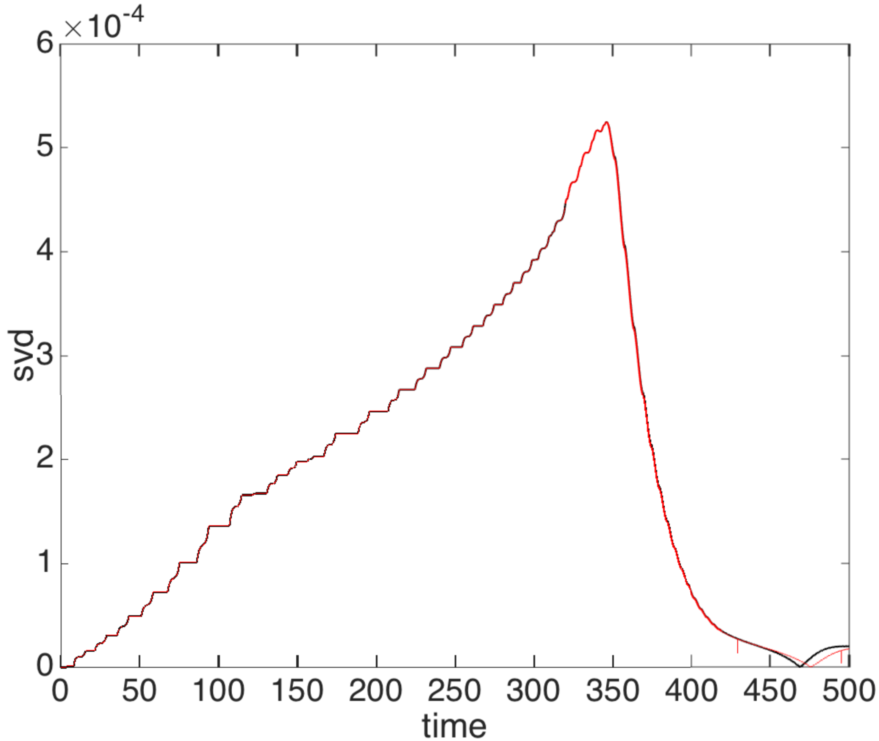

Second order conditions

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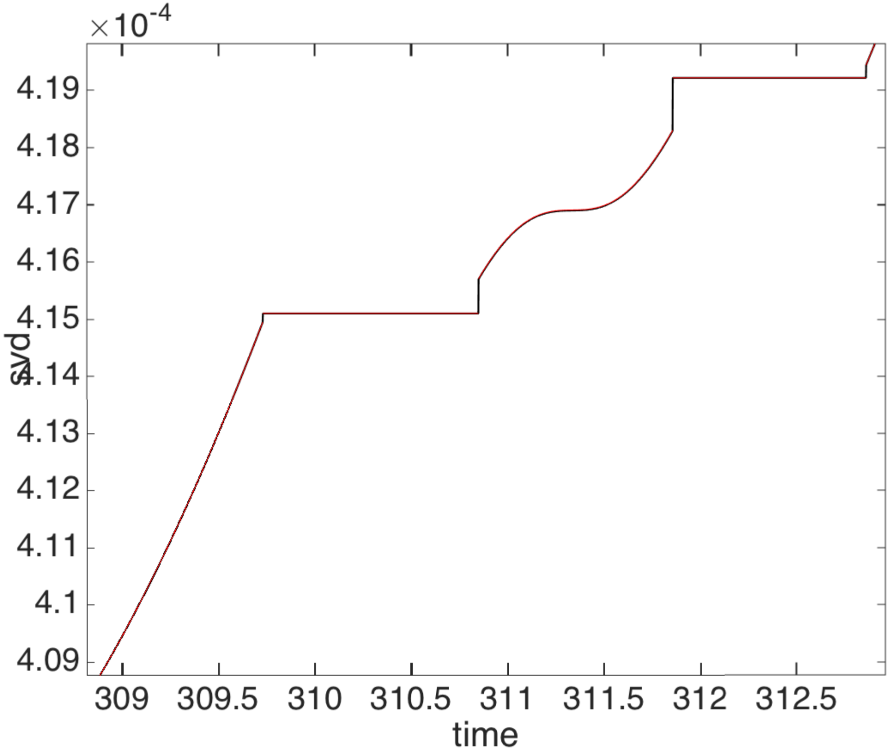

Second order conditions

Chattering.

Prop. (Zelikin'1994) Singular arcs are $\mathscr{C}^0$-locally optimal in short time provided $p_v q < 0$ (inward pointing control). The local synthesis is given by the concatenation of chattering entering the singular arc, singular arc, chattering exiting the singular arc.

Perspectives.

- L^1-minimization (dim ∞): Zero and singular arcs
- Conjugate points at or between switching points
- Result extends to time dependent potential $V(t,q)$ (3BP, see Zelikin & Borisov'2003)
- Chattering: Physical significance? Complete optimality analysis? Approximation (BV regularization, Ghezzi'2014)?

References

[1] Differential pathfollowing for regular optimal control problems. *Optim. Methods Softw.* **27** (2012), no. 2, 177–196. apo.enseeiht.fr/hampath (with Cots, O.; Gergaud, J.)

[2] Minimum time control of the restricted three-body problem. *SIAM J. Control Optim.* **50** (2012), no. 6, 3178–3202 (with Daoud, B.)

[3] Minimum fuel control of the planar circular restricted three-body problem. *Celestial Mech. Dynam. Astronom.* **114** (2012), no. 1, 137–150 (with Daoud, B.; Gergaud, J.)

Special Radon Semester

New Trends in Calculus of Variations

Oct. 13th - Dec. 12th 2014

Location · RICAM · Linz · Austria

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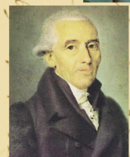

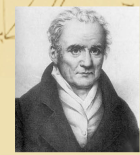

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A special semester will take place at RICAM in Linz (fall 2014) on the topic of "Calculus of Variations". This very active field focuses on optimizing mathematical functionals and enables the development of methods applied to a wide variety of problems. The semester is organized into four workshops and two schools.
The workshops "Image Analysis" and "Optimal Transportation" will be preceded by two fall schools intended for non-specialists, mainly students. Theoretical, numerical and applied aspects will be addressed during the event which will bring together experts in the field and young researchers including many PhD students.

Workshops

"Shape Optimization" October 13th—17th - Organizers: E.Oudet , M. Rumpf

The optimization of geometry and topology of shapes requires the combination of a variety of mathematical tools among them different implicit shape representations, relaxation theory, homogenization, duality techniques, and optimal transportation methods. The workshop will bring together experts on geometry, regularity theory, structural mechanics, numerical analysis, and optimization to discuss recent trends, identify synergies between different disciplines, and explore new directions.

Invited Speakers include: G. Allaire, S. Bartels, D. Bucur, B. Bourdin, G. Buttazzo, S. Esedoglu, H. Harbrecht, A. Henrot, M. Hintermüller, S. Masnou, M. Stingl, B. Wirth

"Variational Methods in Imaging" October 27th—31st- Organizers: G. Peyré, C. Schnörr

The workshop "Variational methods in imaging" is composed of invited lectures given by experts in the field of imaging sciences. It aims at presenting an interplay between research in applied mathematics (PDEs, optimization, inverse problems, optimal transport, shape spaces) and applications in imaging (image processing, computer vision, computer graphics, computational anatomy)

Invited Speakers include: J-F. Aujol, M. Bauer, K. Bredies, M. Burger, A. Chambolle, E. Chouzenoux, D. Cremers, J. Fadili, M. Figueiredo, G. Gilboa M. Hintermüller, S. Joshi, R. Kimmel, P. Micho, M. Nikolova, J.-C. Pesquet, T. Pock, M. Rumpf, C. Schoenlieb, A. Srivastava, G. Steidl, A.Trouvé, J. Weickert

"Geometric Control and Related Fields" November 17th—21st- Organizers: J.B. Caillau, T. Haberkorn

This workshop intends to gather people from geometric control and related fields including Riemannian geometry and generalizations, Hamilton-Jacobi theory, celestial mechanics, quantum control, algebraic and numerical methods for ODEs and control. Young researchers are especially welcome to participate.

Invited Speakers include: B. Bonnard, F. Diacu, J. Fejoz , M. Grochowski, L. Guijarro, D. Henrion, A. Jorba, W. Krynski, A. Menucci, M. Mirrahimi, L. Rizzi, G. Stefani, H. Zidani

"Optimal Transport" December 8th-12th- Organizers: G. Carlier, T. Champion, F. Santambrogio

Optimal transport is the variational theory that looks at how to displace masses at minimal-cost and how to choose optimal paths and pairings This workshop will focus on its applications and connections with applied sciences, and will gather world-leading experts, as well as young researchers, actively working in modeling, PDEs, numerical methods, with an emphasis on the applications to economics, social sciences, biology, physics, image processing and many others.

Schools

"Variational Methods in Imaging" October 24th—27th - Organizer: O. Scherzer

Lecturers: A. Chambolle, S. Arridge.

"Optimal Transportation" December 2nd-5th — Organizers: G. Carlier, T. Champion, F. Santambrogio

The thematic school will focus on the applied side the theory: numerical methods, applications to evolution PDEs, and multi-marginal problems with applications to physics and economics.

Lecturers: J.-D. Benamou, J.-A. Carrillo, L. De Pascale

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