Optimal control problems on stratifiable state constraints sets.

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informatics mathematics

ITN SADCO
Initial Training Network
Sensitivity Analysis for Deterministic Controller Design

Introduction

We consider an infinite horizon problem with state constraints $\mathcal{K}$:

$$\text{(P)} \quad \inf \left\{ \int_0^\infty e^{-\lambda t} \ell(y_x^u(t), u(t))dt \;\middle|\; \begin{array}{c} u : [0, +\infty) \to \mathcal{A} \text{ measurable} \\ y_x^u(t) \in \mathcal{K} \quad \forall t \geq 0 \end{array} \right\}.$$

where $\lambda > 0$ is fixed and $y_x^u(\cdot)$ is a trajectory of the control system

$$\begin{cases} \dot{y} = f(y, u) & \text{a.e. } t \geq 0 \\ y(0) = x \in \mathcal{K} \end{cases}$$

We are mainly concerned with a characterization of the Value Function of (P) as the bilateral solution to a Hamilton-Jacobi-Bellman (HJB) equation

$$\lambda\vartheta + H(x, \nabla\vartheta(x)) = 0.$$

Introduction

Basic properties of the Value Function

$$\vartheta(x) := \inf_{u \in \mathbb{A}(x)} \int_0^\infty e^{-\lambda t} \ell(y_x^u(t), u(t))dt, \qquad \forall x \in \mathcal{K},$$

where $\mathbb{A}(x) = \{u : [0, +\infty) \to \mathcal{A} \text{ measurable } \mid y_x^u(t) \in \mathcal{K} \quad \forall t \geq 0\}$.

Known facts (Standard hypothesis + Convexity of dynamics)

If $\lambda > \lambda_0$ for some $\lambda_0 > 0$:

- $\forall x \in \mathcal{K}$, $\exists u^* \in \mathbb{A}(x)$ a minimizer provided $\vartheta(x) \in \mathbb{R}$.
- $\vartheta : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous.
- ϑ is the unique function that satisfies the DPP.
- ϑ is a supersolution of the HJB equation on $\mathcal{K}$.
- ϑ is a subsolution of the HJB equation on $\mathrm{int}(\mathcal{K})$.

Introduction

More basic properties of the Value Function

$$\vartheta(x) := \inf_{u \in \mathbb{A}(x)} \int_0^\infty e^{-\lambda t} \ell(y_x^u(t), u(t))dt, \qquad \forall x \in \mathcal{K}.$$

Known facts (Comparison principle)

- if φ is a supersolution of the HJB equation on $\mathcal{K}$ then

$$\vartheta(x) \leq \varphi(x), \quad \forall x \in \mathcal{K}.$$

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Known facts (Comparison principle)

- if φ is a supersolution of the HJB equation on $\mathcal{K}$ then

$$\vartheta(x) \leq \varphi(x), \quad \forall x \in \mathcal{K}.$$

- if φ is a subsolution of the HJB equation on $\mathrm{int}(\mathcal{K})$ then

$$\vartheta(x) \geq \varphi(x), \quad \forall x \in \mathcal{K} \qquad \longleftarrow \text{May not be true!}.$$

Feasible Neighboring Trajectories Approach

Soner, Frankowska-Vinter, Clarke-Stern, among many others

When it does hold ? ?

- $\vartheta(\cdot)$ is continuous up to the boundary.
- Interior approximation of trajectories.
- Some monotonicity properties along trajectories.

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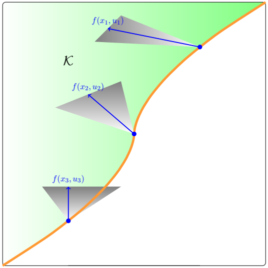
What do we need to achieve one of these ? ?

$\looparrowright$ Tameness properties : $\mathcal{K} = \overline{\mathrm{int}(\mathcal{K})}$.

$\looparrowright$ Pointing Conditions

(Inward or Outward).



Example

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ u \end{pmatrix}$$

$$y_1(0) = x,$$

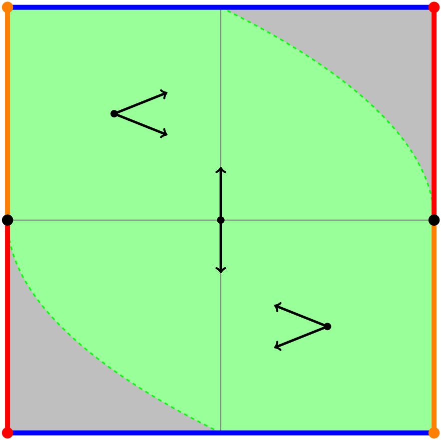
$$y_2(0) = v,$$

$$u \in \mathcal{A} := [-1, 1]$$

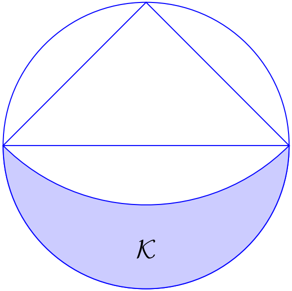
$$y_1(t), y_2(t) \in [-r, r]$$

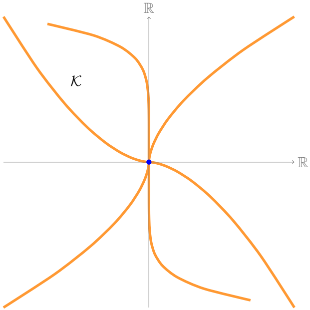
Note that :

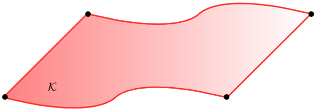
$$\left\langle \begin{pmatrix} 0 \\ u \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle = 0$$

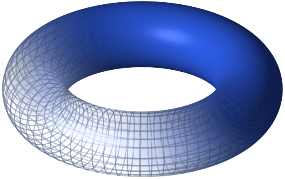


More complicated state constraints









Dynamic Programming Principle

Recall

$$\vartheta(x) = \inf_{u \in \mathbb{A}(x)} \left\{ \int_0^T e^{-\lambda t} \ell(y_x^u(t), u(t)) dt + e^{-\lambda T} \vartheta(y_x^u(T)) \right\}, \quad \forall T \geq 0.$$

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In particular, the Value Function is strongly increasing :

Definition

A function $\varphi : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ is called strongly increasing provided $\mathrm{dom}\,\mathbb{A} \subseteq \mathrm{dom}\,\varphi$ and $\forall x \in \mathcal{K}$, $\forall u \in \mathbb{A}(x)$

$$\varphi(x) \leq e^{-\lambda t}\varphi(y_x^u(t)) + \int_0^t e^{-\lambda s}\ell(y_x^u(s), u(s))ds \quad \forall t \geq 0.$$

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$$\varphi(x) \leq e^{-\lambda t} \varphi(y_x^u(t)) + \int_0^t e^{-\lambda s} \ell(y_x^u(s), u(s))ds \quad \forall t \geq 0.$$

Under suitable assumptions :

$$\varphi \text{ is strongly increasing} \quad \Rightarrow \quad \vartheta(x) \geq \varphi(x), \ \forall x \in \mathcal{K}.$$

Standing Hypothesis (SH)

- $\mathcal{K}$ is closed and $\mathcal{A} \subseteq \mathbb{R}^m$ is nonempty and compact.
- $\ell : \mathbb{R}^N \times \mathcal{A} \to [0, +\infty)$ and $f : \mathbb{R}^N \times \mathcal{A} \to \mathbb{R}^N$ are continuous and Lipschitz w.r.t. the state :

$$\exists L > 0 \text{ such that } \quad \left.\begin{array}{c} |f(x,u) - f(y,u)| \\ |\ell(x,u) - \ell(y,u)| \end{array}\right\} \leq L|x-y| \qquad \forall u \in \mathcal{A}.$$

- The cost has linear growth w.r.t. the state :

$$\exists c > 0 \text{ such that } \quad \ell(x,u) \leq c(1+|x|) \qquad \forall u \in \mathcal{A}.$$

- We assume convexity of the augmented dynamics :

$$\left\{ \begin{pmatrix} f(x,u) \\ e^{-\lambda t}(\ell(x,u)+r) \end{pmatrix} \,\middle|\, \begin{array}{c} u \in \mathcal{A}, \;\; r \geq 0 \\ r \leq c(1+|x|) - \ell(x,u) \end{array} \right\}, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

Example

$\mathcal{A}$ convex, $f(x,u) = f_0(x) + F(x)u$ and $u \mapsto \ell(x,u)$ convex.

Definition (Subdifferentials)

Let $\varphi : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function. A vector $\zeta \in \mathbb{R}^N$ is called a viscosity subgradient of φ at $x \in \operatorname{dom} \varphi$ if and only if :

$$\exists g \in \mathcal{C}^1(\mathbb{R}^N) \text{ s.t. } \nabla g(x) = \zeta \text{ and } \varphi - g \text{ attains a local minimum at } x.$$

We denote by $\partial_V \varphi(x)$ the set of all viscosity subgradients of φ at x.

Proposition

Suppose that (SH) holds and let $\varphi : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function. If φ is strongly increasing then

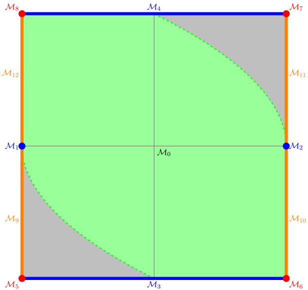
$$\lambda\varphi(x) + H(x, \zeta) \leq 0 \qquad \forall x \in \operatorname{int}(\mathcal{K}),\ \forall \zeta \in \partial_V \varphi(x).$$

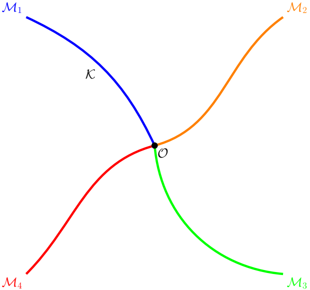
Stratifiable sets

Definition

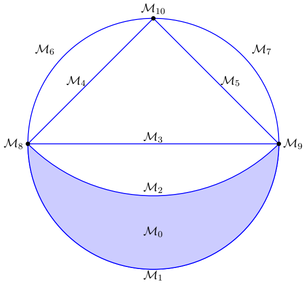
A closed set $\mathcal{K} \subseteq \mathbb{R}^N$ *is said to be* *stratifiable* *if there exists a locally finite collection* $\{\mathcal{M}_i : i \in \mathcal{I}\}$ *of embedded manifolds of* $\mathbb{R}^N$ *such that :*

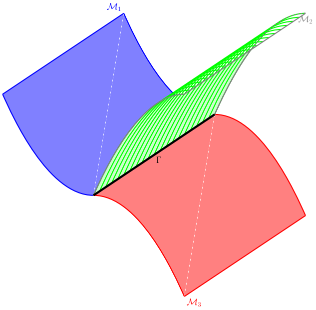
- $\mathcal{K} = \bigcup_{i\in\mathcal{I}} \mathcal{M}_i$ *and* $\mathcal{M}_i \cap \mathcal{M}_j = \emptyset$ *whenever* $i \neq j$.
- $\mathcal{M}_i \cap \overline{\mathcal{M}_j} \neq \emptyset$, *then* $\mathcal{M}_i \subseteq \overline{\mathcal{M}_j}$ *and* $\dim(\mathcal{M}_i) < \dim(\mathcal{M}_j)$.





Stratifiable sets





The class of stratifiable sets on $\mathbb{R}^N$ is wide, it includes :

- Semilinear sets $\rightarrow$ finite union of open polyhedra.
- Semialgebraic sets $\rightarrow$ finite union of polynomial manifolds.
- Subanalytic sets $\rightarrow$ locally finite union of analytic manifolds.

Characterization of subsolutions

Some basic definitions and notation

Assume that $\mathcal{K}$ is stratifiable and let $\{\mathcal{M}_i\}$ be its strata.
For each stratum $\mathcal{M}_i$ we define

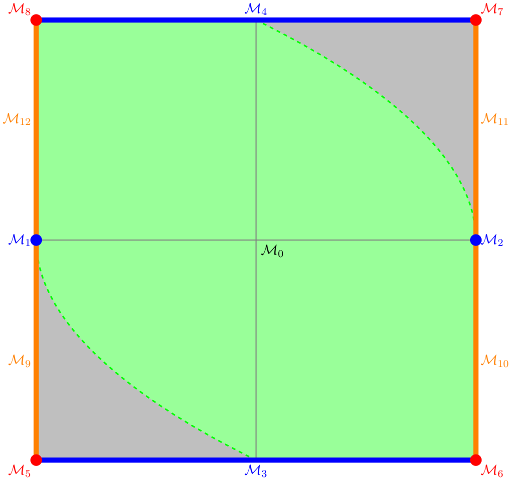
- the set of tangent controls as the map $\mathcal{A}_i : \mathcal{M}_i \rightrightarrows \mathcal{A}$ given by

$$\mathcal{A}_i(x) := \{u \in \mathcal{A} \mid f(x,u) \in \mathcal{T}_{\mathcal{M}_i}(x)\}.$$

- the tangential Hamiltonian as the map $H_i : \mathcal{M}_i \times \mathbb{R}^N \to \mathbb{R}$ given by

$$H_i(x,\zeta) = \max_{u \in \mathcal{A}_i(x)} \{-\langle \zeta, f(x,u)\rangle - \ell(x,u)\}.$$

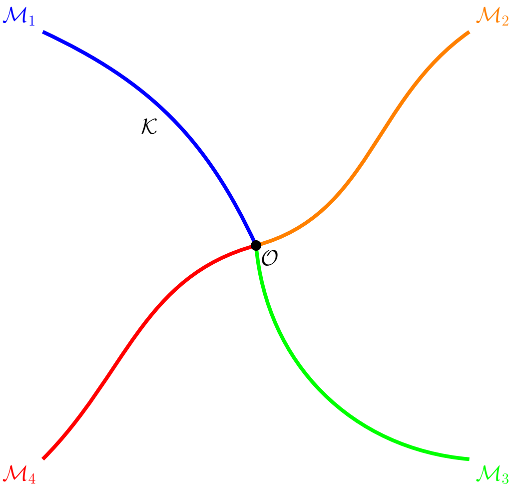
Tangent controls



$$f(x_1, x_2, u) = (x_2, u)$$
$$u \in [-1, 1]$$

$$\mathcal{A}_0(x) = \mathcal{A}$$
$$\mathcal{A}_i(x) = \{0\} \text{ for } i = 1, \ldots, 4$$
$$\mathcal{A}_j(x) = \emptyset \text{ for } j = 5, \ldots, 12$$

Tangent controls



$\exists A_i \subseteq \mathcal{A}$ such that

$$f(x, \mathcal{A}) \cap \mathcal{T}_{\mathcal{M}_i}(x) = f(x, \mathcal{A}_i).$$

$$\mathcal{A}_i(x) = \mathcal{A}_i, \quad \forall i = 1, \dots, 4$$

Let $\mathcal{M}_0 = \{\mathcal{O}\}$:

$$\mathcal{A}_0(\mathcal{O}) = \{u \in \mathcal{A} \mid f(\mathcal{O}, u) = 0\}.$$

Characterization of subsolutions (necessity)

Proposition (CH - Zidani)

Suppose that (SH) holds with $\mathcal{K}$ is stratifiable and let $\varphi : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function. Assume that

(H_0) $\quad$ *$\mathcal{A}_i$ is a Lipschitz set-valued map or has empty images on $\mathcal{M}_i$.*

If φ is strongly increasing then for each $i \in \mathcal{I}$

$$(\bigstar) \qquad \lambda\varphi(x) + H_i(x, \zeta) \leq 0 \qquad \forall x \in \mathcal{M}_i,\ \forall \zeta \in \partial_V \varphi_i(x),$$

where $\varphi_i(x) = \varphi(x)$ if $x \in \overline{\mathcal{M}}_i$ and $+\infty$ otherwise.

Remark

$(\bigstar)$ is equivalent to say : $\forall i \in \mathcal{I}$, $\forall x \in \mathcal{M}_i$ and $\forall g \in \mathcal{C}^1(\mathbb{R}^N)$ such that $\varphi - g$ attains a local minimum at x relative to $\mathcal{M}_i$

$$\lambda\varphi(x) + H_i(x, \nabla g(x)) \leq 0.$$

The converse ?

Strong Invariance Principle on $\mathrm{int}(\mathcal{K})$

$$\lambda\varphi(x) + \max_{u\in\mathcal{A}} \{-\langle\zeta, f(x,u)\rangle - \ell(x,u)\} \leq 0 \qquad \forall x \in \mathrm{int}(\mathcal{K}),\ \forall\zeta \in \partial_V\varphi(x).$$

$$\Downarrow$$

$y_x^u(t) \in \mathrm{int}(\mathcal{K})$ for every $t \in (a,b)$, where $0 \leq a < b < +\infty$ then

$$\varphi(y_x^u(a)) \leq e^{-\lambda(b-a)}\varphi(y_x^u(b)) + e^{\lambda a}\int_a^b e^{-\lambda t}\ell(y_x^u(t),u(t))dt.$$

$$\Downarrow$$

Any admissible trajectory y_x^u defined on $[0,T]$ with $y_x^u(t) \in \mathrm{int}(\mathcal{K})$ for every $t \in (0,T)$ satisfies the Strong Increasing Inequality.

The converse ?

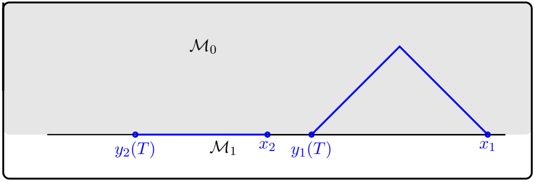
Strong Invariance Principle on each stratum

$$\lambda\varphi(x) + \max_{u\in\mathcal{A}_i(x)} \{-\langle\zeta, f(x,u)\rangle - \ell(x,u)\} \leq 0 \qquad \forall x \in \mathcal{M}_i,\ \forall\zeta \in \partial_V\varphi_i(x).$$

$$\Downarrow$$

$y_x^u(t) \in \mathcal{M}_i$ for every $t \in (a,b)$, where $0 \leq a < b < +\infty$ then

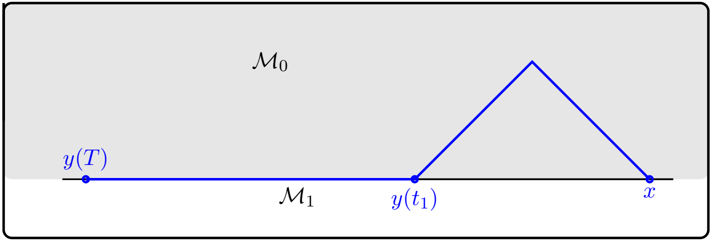
$$\varphi(y_x^u(a)) \leq e^{-\lambda(b-a)}\varphi(y_x^u(b)) + e^{\lambda a}\int_a^b e^{-\lambda t}\ell(y_x^u(t), u(t))dt.$$



Suppose that $y_x^u(t) \in \mathcal{K}$, $\forall t \in [0, T]$ and there exists a partition of $[0, T]$

$$\pi = \{0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = T\}$$

so that $\forall l \in \{0, \ldots, n\}, \exists \mathcal{M}_i$ with $y_x^u(t) \in \mathcal{M}_i, \forall t \in (t_l, t_{l+1})$.



Whence, $\forall l \in \{0, \ldots, n\}$ we have

$$\varphi(y_x^u(t_l)) \leq e^{-\lambda(t_{l+1}-t_l)}\varphi(y_x^u(t_{l+1})) + e^{\lambda t_l} \int_{t_l}^{t_{l+1}} e^{-\lambda t} \ell(y_x^u(t), u(t))dt.$$

Therefore y_x^u satisfies the Strong Increasing Inequality ! !

Suppose that $y_x^u(t) \in \mathcal{K}$, $\forall t \in [0, T]$ and there exists a partition of $[0, T]$

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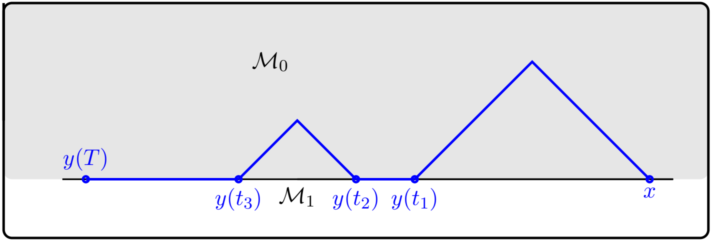

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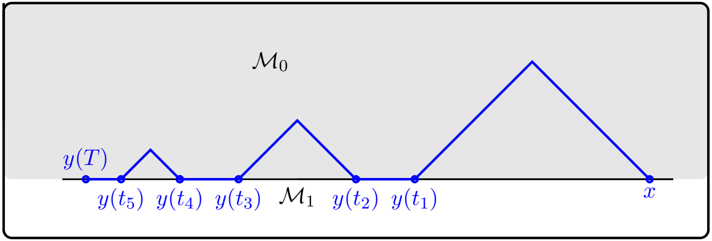

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Suppose that $y_x^u(t) \in \mathcal{K}$, $\forall t \in [0, T]$ and there is NO partition of $[0, T]$

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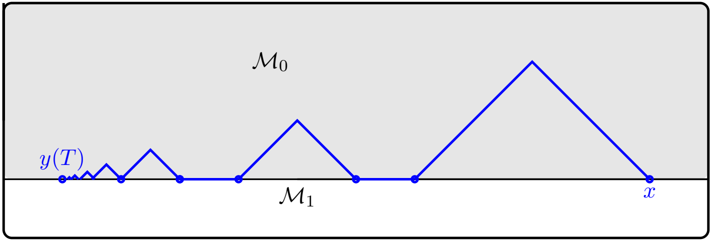
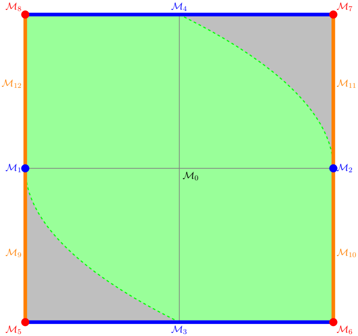


What if the trajectory "chatters" between two or more strata ? ? ?

Controllability assumption

$$(H_1) \quad \begin{cases} \forall i \in \mathcal{I} \text{ with } \operatorname{dom} \mathcal{A}_i \neq \emptyset,\ \exists \delta_i, \Delta_i > 0 \text{ such that} \\ \mathcal{R}^{(t)}(x) \cap \overline{\mathcal{M}}_i \subseteq \displaystyle\bigcup_{s \in [0, \Delta_i t]} \mathcal{R}_i^{(s)}(x), \quad \forall x \in \mathcal{M}_i,\ \forall t \in [0, \delta_i]. \end{cases}$$

$\mathcal{R}^{(t)}(\cdot)$: reachable set at time t of $x \mapsto f(x, \mathcal{A})$.
$\mathcal{R}_i^{(t)}(\cdot)$: reachable set at time t of $x \mapsto f(x, \mathcal{A}_i(x))$.



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Lemma

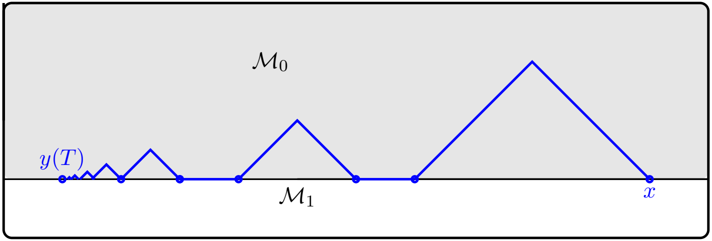
Suppose (SH) with $\mathcal{K}$ stratifiable. Assume that (H_0) and (H_1) hold. For any $x \in \mathcal{K}$ and $T > 0$ there exists $L > 0$ such that : $\forall u \in \mathbb{A}(x), \forall \varepsilon > 0$ if $y_x^u(b), y_x^u(a) \in \mathcal{M}_i$ with $0 \leq a < b \leq T$ for some $i \in \mathcal{I}$ then,

$$\varphi(y_x^u(a)) \leq e^{\lambda \varepsilon} \left(e^{-\lambda(b-a)} \varphi(y_x^u(b)) + e^{\lambda a} \int_a^b e^{-\lambda t} \ell(y_x^u, u) dt \right) + L\varepsilon.$$

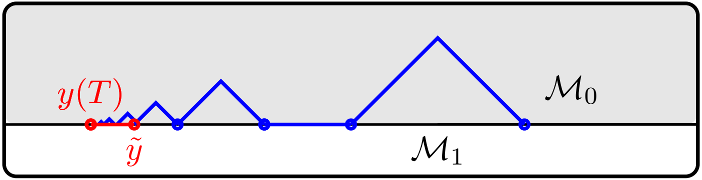
provided $\quad \operatorname{meas}(\{t \in [a, b] \mid y_x^u(t) \notin \mathcal{M}_i\}) < \varepsilon.$

Controllability assumption :

Chattering trajectory :



Approximated trajectory :



Characterization of subsolutions (sufficiency)

Proposition (CH - Zidani)

Suppose that (SH) with $\mathcal{K}$ stratifiable. Assume that (H_0) and (H_1) hold. Let $\varphi : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function with $\mathrm{dom}\,\mathbb{A} \subseteq \mathrm{dom}\,\varphi$ that satisfies

$$(\bigstar) \qquad \lambda\varphi(x) + H_i(x,\zeta) \leq 0 \qquad \forall x \in \mathcal{M}_i,\ \forall \zeta \in \partial_V \varphi_i(x),\ \forall i \in \mathcal{I}.$$

Then φ is strongly increasing.

Recall

$(\bigstar)$ is equivalent to say : $\forall i \in \mathcal{I}$, $\forall x \in \mathcal{M}_i$ and $\forall g \in \mathcal{C}^1(\mathbb{R}^N)$ such that $\varphi - g$ attains a local minimum at x relative to $\mathcal{M}_i$

$$\lambda\varphi(x) + H_i(x, \nabla g(x)) \leq 0.$$

Characterization of the Value Function

Theorem (CH - Zidani)

Suppose (SH) with $\mathcal{K}$ stratifiable and $\lambda > \lambda_0$. Assume (H_0) and (H_1). Then the Value Function

$$\vartheta(x) := \inf \left\{ \int_0^\infty e^{-\lambda t} \ell(y_x^u(t), u(t)) dt \;\middle|\; u \in \mathbb{A}(x) \right\}.$$

is the unique l.s.c. function with linear growth which is $+\infty$ on $\mathbb{R}^N \setminus \mathcal{K}$ and that satisfies :

$$\begin{aligned} \lambda\vartheta(x) + H(x,\zeta) \geq 0 \qquad & \forall x \in \mathcal{K},\ \forall \zeta \in \partial_V \vartheta(x), \\ \lambda\vartheta(x) + H_i(x,\zeta) \leq 0 \qquad & \forall x \in \mathcal{M}_i,\ \forall \zeta \in \partial_V \vartheta_i(x),\ \forall i \in \mathcal{I}, \end{aligned}$$

where $\vartheta_i(x) = \vartheta(x)$ if $x \in \overline{\mathcal{M}_i}$ and $+\infty$ otherwise.

Characterization of the Value Function : $\mathcal{M}_0 = \text{int}(\mathcal{K})$.

Theorem (CH - Zidani)

Suppose (SH) with $\mathcal{K}$ stratifiable with $\text{int}(\mathcal{K}) \neq \emptyset$ and $\lambda > \lambda_0$. Assume (H_0) and (H_1). Then the Value Function

$$\vartheta(x) := \inf \left\{ \int_0^\infty e^{-\lambda t} \ell(y_x^u(t), u(t)) dt \;\middle|\; u \in \mathbb{A}(x) \right\}.$$

is the unique l.s.c. function with linear growth which is $+\infty$ on $\mathbb{R}^N \setminus \mathcal{K}$ and that satisfies :

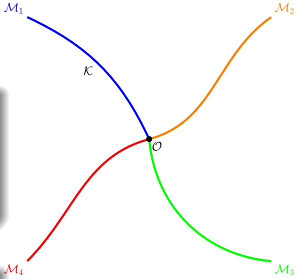
$$\begin{aligned}
\lambda\vartheta(x) + H(x,\zeta) &\geq 0 && \forall x \in \mathcal{K},\ \forall \zeta \in \partial_V \vartheta(x), \\
\lambda\vartheta(x) + H(x,\zeta) &\leq 0 && \forall x \in \text{int}(\mathcal{K}),\ \forall \zeta \in \partial_V \vartheta(x), \\
\lambda\vartheta(x) + H_i(x,\zeta) &\leq 0 && \forall x \in \mathcal{M}_i,\ \forall \zeta \in \partial_V \vartheta_i(x),\ \forall i \in \mathcal{I} \setminus \{0\},
\end{aligned}$$

where $\vartheta_i(x) = \vartheta(x)$ if $x \in \overline{\mathcal{M}_i}$ and $+\infty$ otherwise.

Applications to Networks

Suppose $\mathcal{K}$ is a network with one junction $\mathcal{O}$ and let $\mathcal{M}_1, \ldots, \mathcal{M}_p$ be its branches.
Assume that for each $i \in \{1, \ldots, p\}$, $\exists \mathcal{A}_i \subseteq \mathcal{A}$ s.t.

$$(H_3) \qquad f(x, \mathcal{A}) \cap \mathcal{T}_{\mathcal{M}_i}(x) = f(x, \mathcal{A}_i), \quad \forall x \in \mathcal{M}_i.$$



For instance, for some $v_i \in \mathbb{R}^N \setminus \{0\}$

$$\mathcal{M}_i = (0, +\infty)v_i \text{ and } f(x, u) = f_i(x, u)v_i, \quad \forall x \in \mathcal{M}_i$$

with and f_i real-valued.

Claim

The hypothesis (H_0) and (H_1) are satisfied.

Applications to Networks

Theorem (CH - Zidani)

Suppose that (SH) with $\mathcal{K}$ a network as before and $\lambda > \lambda_0$. Assume that (H_3) holds and let

$$\mathcal{A}_0 = \{u \in \mathcal{A} \mid \; f(\mathcal{O}, u) = 0\}.$$

Then the Value Function is the unique l.s.c. function with linear growth which is $+\infty$ on $\mathbb{R}^N \setminus \mathcal{K}$ and that satisfies :

$$\lambda\vartheta(x) + \max_{u\in\mathcal{A}_i}\{-\langle\zeta, f(x,u)\rangle - \ell(x,u)\} = 0 \quad \forall x \in \mathcal{M}_i,\ \forall\zeta \in \partial_V\vartheta(x),$$

$$\lambda\vartheta(\mathcal{O}) + H(\mathcal{O},\zeta) \geq 0 \quad \forall\zeta \in \partial_V\vartheta(\mathcal{O}),$$

$$\lambda\vartheta(\mathcal{O}) - \min_{u\in\mathcal{A}_0}\ell(\mathcal{O},u) \leq 0.$$

Final remarks

- The interior of $\mathcal{K}$ can always be taken as a stratum and so, the constrained Hamiton-Jacobi equation proposed by Soner is included in the set of equations proposed in the main theorem.
- The characterize of the Value Function neither requires its continuity nor that the state constraint set has empty interior.
- Under the continuity of the Value Function (on its domain) the controllability assumption can be dropped.
- The controllability assumption can be replaced by a stronger but easier to verify hypothesis of full controllability on manifolds.
- The characterization for networks can be extended to a suitable notion of generalized network where the junction is replaced by a manifold.

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Thanks for your attention !