The Tschebychev Scalarization Method for Solving Multi-Objective Optimal Control Problems

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Outline

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1. Multi-Objective Optimal Control problem

2. The Pareto Front

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3. Scalarization : Weighted-Sum and Tschebychev Approach

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4. Example 1: Tumour Anti-Angiogenesis

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4. Example 1: Tumour Anti-Angiogenesis
5. Example 2: Fed-Batch Bioreactor

Single-Objective Optimal Control Problem

$$
(\text{OCP}) \quad \begin{cases}
\min & \varphi_1(x(t_f), t_f) \\
\text{s.t.} & \dot{x}(t) = f(x(t), u(t), t)\,, \quad \text{a.e. } t \in [0, t_f]\,, \\
& \phi(x(0), x(t_f), t_f) = 0\,, \\
& \widetilde{\phi}(x(0), x(t_f), t_f) \le 0\,, \\
& C(x(t), u(t), t) \le 0\,, \quad \text{a.e. } t \in [0, t_f]\,, \\
& S(x(t), t) \le 0\,, \quad \text{all } t \in [0, t_f]\,,
\end{cases}
$$

state variable $x \in \mathbb{R}^n$, *control* variable $u \in \mathbb{R}^m$,
terminal time t_f is fixed or free.

Multi-Objective Optimal Control Problem

$$
(\text{OCP}_m) \begin{cases} \min & (\varphi_1(x(t_f),t_f),\ldots,\varphi_r(x(t_f),t_f)) \\ \text{s.t.} & \dot{x}(t) = f(x(t),u(t),t)\,, \quad \text{a.e. } t \in [0,t_f]\,, \\ & \phi(x(0),x(t_f),t_f) = 0\,, \\ & \widetilde{\phi}(x(0),x(t_f),t_f) \le 0\,, \\ & C(x(t),u(t),t) \le 0\,, \quad \text{a.e. } t \in [0,t_f]\,, \\ & S(x(t),t) \le 0\,, \quad \text{all } t \in [0,t_f]\,, \end{cases}
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Multi-Objective Optimal Control Problem

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$$

state variable $x \in \mathbb{R}^n$, *control* variable $u \in \mathbb{R}^m$,
terminal time t_f fixed or free.

Assumption: $\varphi_i(x(t_f),t_f) \ge 0$ for all $i = 1,\ldots,r$.

Multi-Objective Optimal Control Problem

The feasible set X consists of triplets

$$(x, u, t_f) \in W^{1,\infty}(0, t_f; \mathbb{R}^n) \times L^{\infty}(0, t_f; \mathbb{R}^n) \times \mathbb{R}_+$$

satisfying the dynamic and terminal constraints as well as the control and state constraints.

Multi-Objective Optimal Control Problem

The feasible triplet (x^*, u^*, t_f^*) is said to be a *Pareto minimum*, if there does not exist a feasible triplet $(x, u, t_f) \in X$ such that

$$\begin{aligned} \varphi_i(x(t_f), t_f)) &\leq \varphi_i(x^*(t_f^*), t_f^*) \quad \text{for all} \quad i = 1, \ldots, r\,. \\ \varphi_k(x(t_f), t_f)) &< \varphi_k(x^*(t_f^*), t_f^*) \quad \text{for one} \quad k \in \{1, \ldots, r\}\,. \end{aligned}$$

Multi-Objective Optimal Control Problem

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On the other hand, (x^*, u^*, t_f^*) is said to be a *weak Pareto minimum*, if there does not exist $(x, u, t_f) \in X$ such that

$$\varphi_i(x(t_f), t_f)) < \varphi_i(x^*(t_f^*), t_f^*) \quad \text{for all } i = 1, \ldots, r\,.$$

The Pareto Front

The set of all objective functional values at the Pareto and weak Pareto minima is said to be the *Pareto front* (or *efficient set*) of Problem (OCP_m) in the objective value space.

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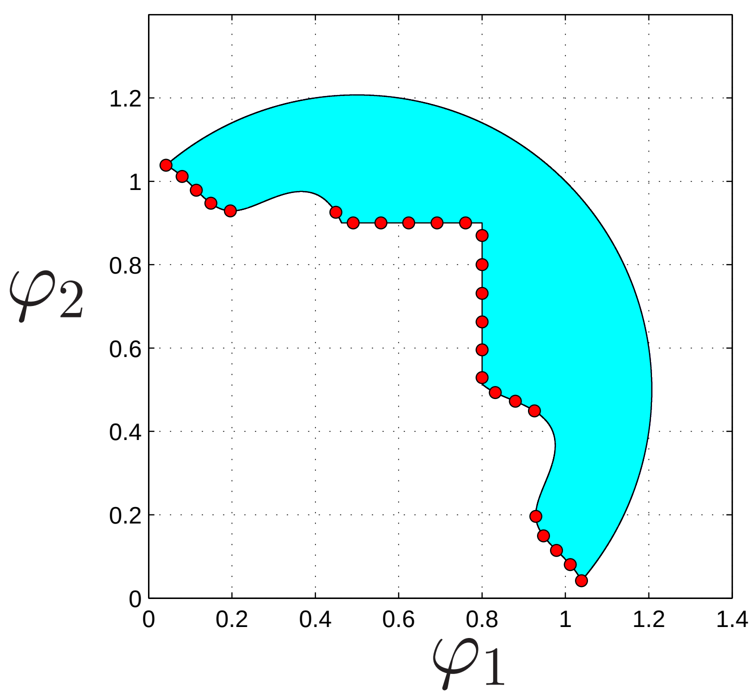
Examples to “generated” Pareto fronts (with $r = 2$):



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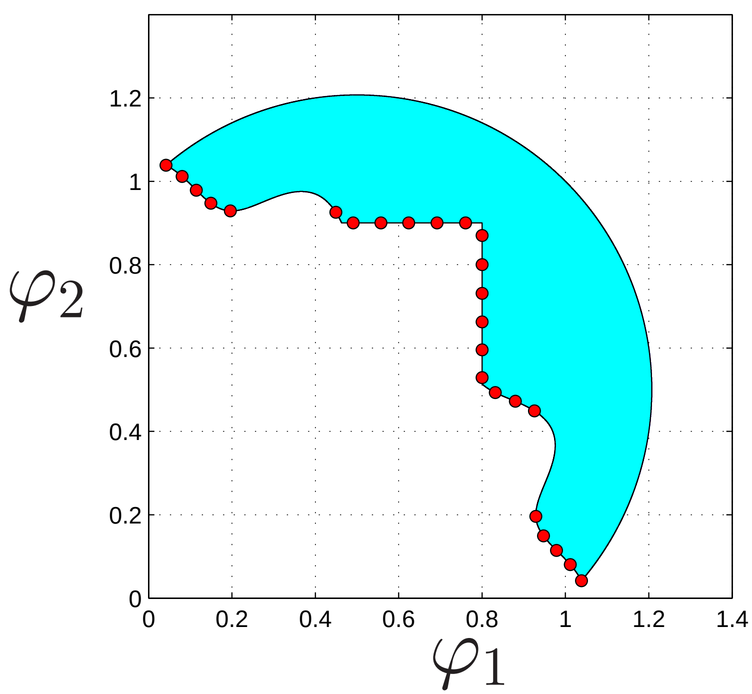
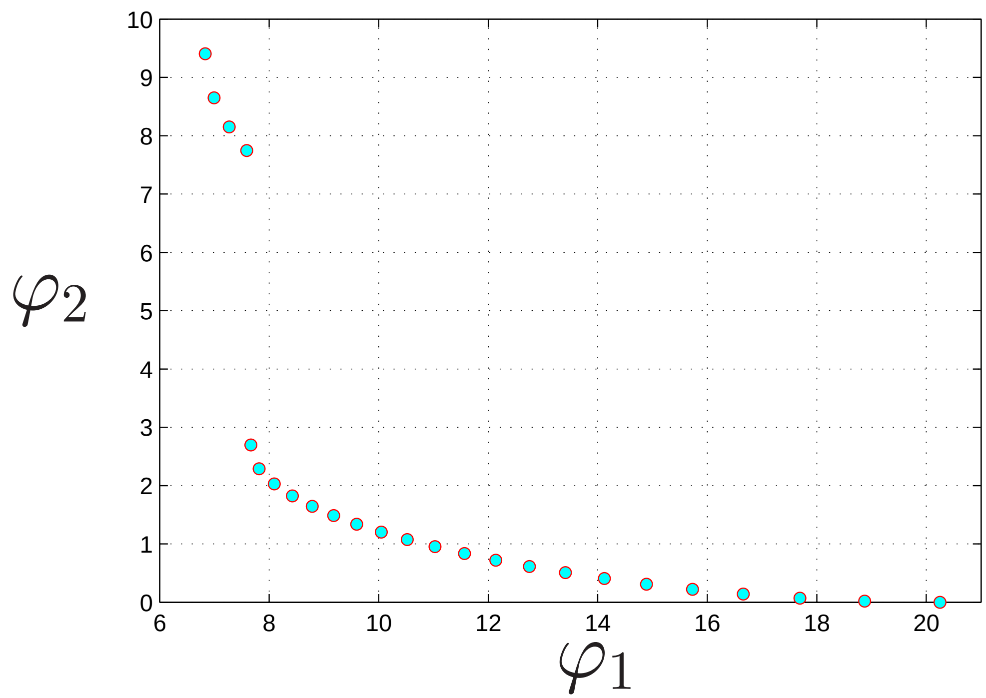




Scalarization

The "popular" *weighted-sum scalarization* :

$$(\mathrm{P}_{ws}) \qquad \min_{(x,u,t_f)\in X} \sum_{i=1}^{r} w_i\, \varphi_i(x(t_f), t_f)\,.$$

where $w_1, \ldots, w_r \geq 0$ are *weights* with $w_1 + \ldots + w_r = 1$.

The *Bolza problem* can equivalently be written in this form.

NOT GOOD FOR NON-CONVEX PROBLEMS WITH A NON-CONVEX PARETO FRONT !

We shall illustrate this on an example.

Scalarization

Weighted Tschebychev problem (*Tschebychev scalarization*) :

$$(\mathbf{P}_w) \qquad \min_{(x,u,t_f)\in X} \max\{w_1\,\varphi_1(x(t_f),t_f),\ldots,w_r\,\varphi_r(x(t_f),t_f)\}\,,$$

where $w_1,\ldots,w_r \geq 0$ are *weights* with $w_1+\ldots+w_r = 1$.

Scalarization

Weighted Tschebychev problem (*Tschebychev scalarization*) :

$$(\mathbf{P}_w) \qquad \min_{(x,u,t_f)\in X} \max\{w_1\,\varphi_1(x(t_f),t_f),\ldots,w_r\,\varphi_r(x(t_f),t_f)\}\,,$$

where $w_1,\ldots,w_r \geq 0$ are *weights* with $w_1+\ldots+w_r = 1$.

Theorem 1 The triplet (x^*,u^*,t_f^*) is a weak Pareto minimum of (OCP_m), if and only if (x^*,u^*,t_f^*) is a solution of (P_w) for some $w_1,\ldots,w_r > 0$.

Scalarization

A smooth re-formulation of (P$_w$):

$$(\text{OCP}_w)\quad \begin{cases} \displaystyle\min_{\substack{\alpha\geq 0\\ (x,u,t_f)\in X}} & \alpha \\ \text{subject to} & w_1\,\varphi_1(x(t_f),t_f)\leq \alpha\,, \\ & \vdots \\ & w_r\,\varphi_r(x(t_f),t_f)\leq \alpha\,. \end{cases}$$

Numerical method: "Discretize then Optimize".

Use Applied Modeling Language AMPL (Fourer et al.) and Interior-Point Optimization Solver IPOPT (Wächter et al.).

Scalarization

Case $r = 2$: $w = w_1 \in [0, 1]$, $w_2 = 1 - w$.

Let $\beta^* = (\beta_1^*, \beta_2^*) \leq 0$ be a so-called utopia point.

Consider the smooth control problem (P_w):

$$(\mathrm{OCP}_w) \quad \begin{cases} \min\limits_{\substack{\alpha \geq 0 \\ (x,u,t_f) \in X}} & \alpha \\ \text{subject to} & w\,(\varphi_1(x(t_f), t_f) - \beta_1^*) \leq \alpha\,, \\ & (1 - w)\,(\varphi_2(x(t_f), t_f) - \beta_2^*) \leq \alpha\,. \end{cases}$$

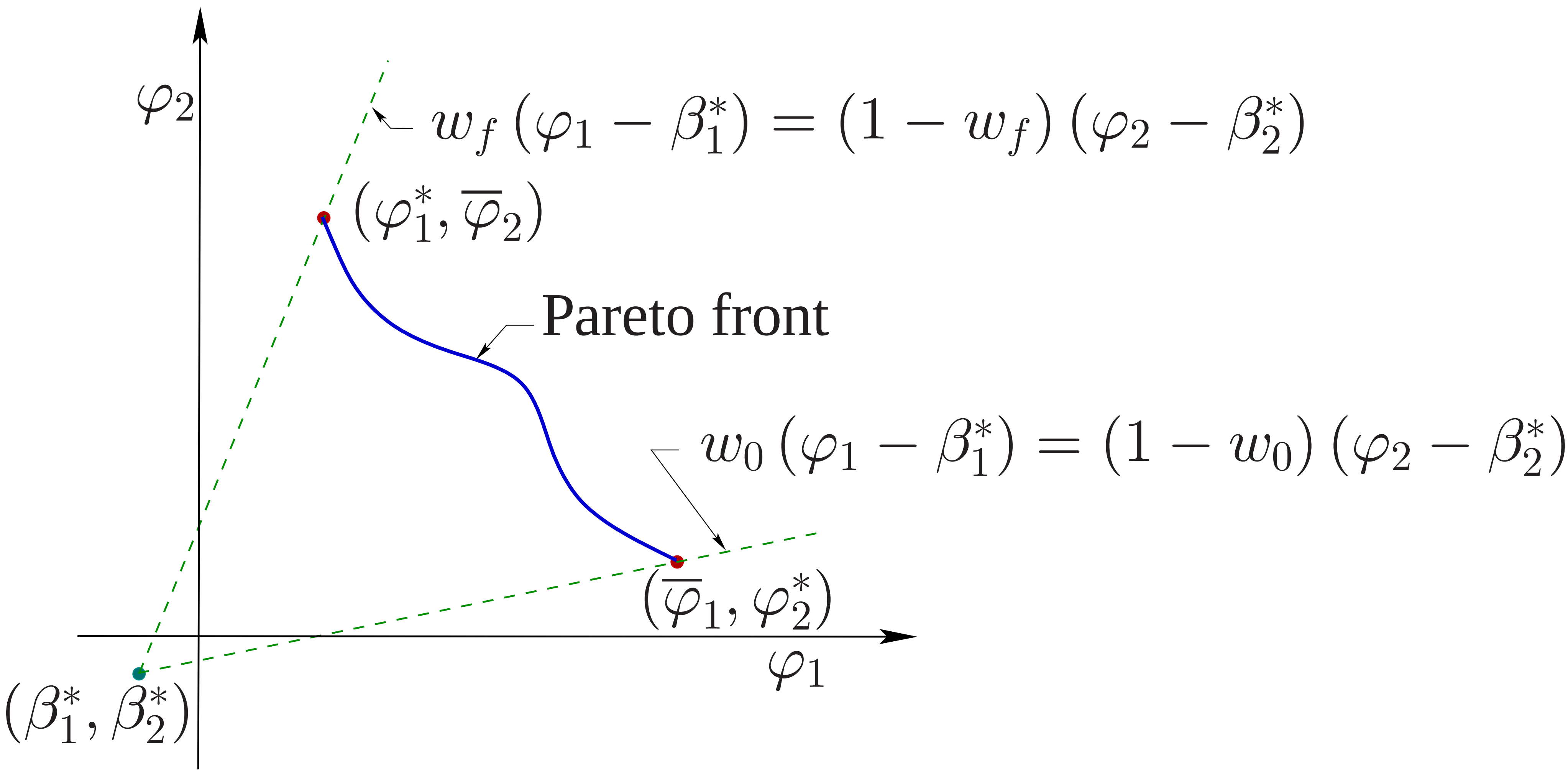
We shall determine a ”meaningful interval”

$$w \in [w_0, w_f]\,, \quad 0 \leq w_0 < w_f \leq 1\,,$$

such that the solution is the same for

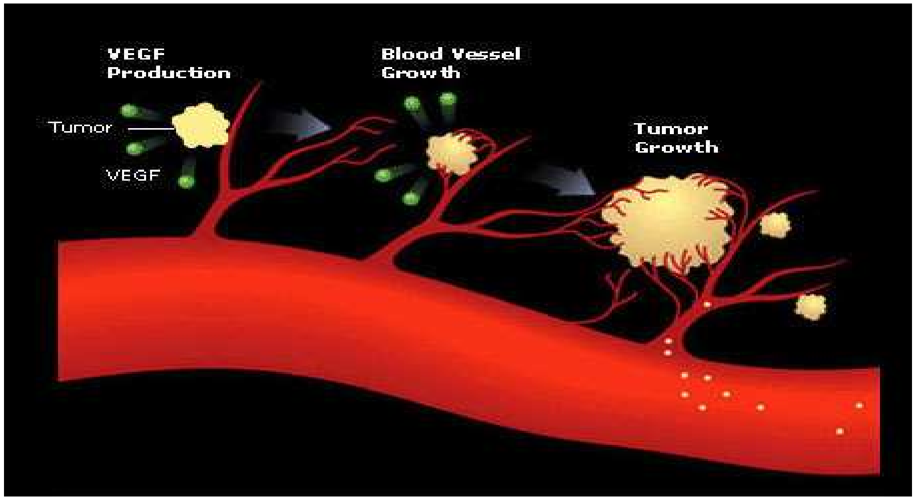
$$w \in [0, w_0] \quad \text{and} \quad w \in [w_f, 1]\,.$$

Boundary weights



Example 1 - Tumour Anti-Angiogenesis

Tumour Anti-Angiogenesis: J. Folkman (1972) et al.



Control model: Ledzewicz, M., Schättler (2007,2011).

State and control variables:

p : primary tumour volume [mm^3]

q : carrying capacity, or endothelial support [mm^3]

u : anti-angiogenic agent

Example 1 - Tumour Anti-angiogenesis

Two competing objective functionals to *minimize*:

- *Final tumour volume*:

$$p(t_f)$$

- Final tumour volume, plus a factor of the *total amount of anti-angiogenic (toxic) agent administered*:

$$p(t_f) + 140 \int_0^{t_f} u(t)\, dt\,.$$

The duration of therapy t_f is free.

Example 1 - Tumour Anti-angiogenesis

$$
\text{(E1)} \quad \left\{ \begin{array}{ll}
\min & (p(t_f),\ p(t_f) + 140\, y(t_f)) \\
\text{s.t.} & \dot{p} = -0.084\, p\, \ln \dfrac{p}{q}, \qquad p(0) = 8000\,, \\
& \dot{q} = 5.85\, q^{2/3} - 0.00873\, q^{4/3} - 0.02\, q - 0.15\, q\, u\,, \\
& \qquad\qquad q(0) = 10000\,, \\
& \dot{y} = u\,, \qquad y(0) = 0\,, \\
& y(t_f) \le 45\,, \quad 0 \le u(t) \le 15\,.
\end{array} \right.
$$

Control u appears linearly: bang-bang and singular arcs.

Example 1 - Tumour Anti-angiogenesis

Scalarization of Problem (E1): For a fixed $w \in [0, 1]$, solve

$$
(\mathrm{E1}_w)\quad \left\{
\begin{array}{ll}
\min & \alpha \\
\text{s.t.} & \dot{p} = -0.084\, p \ln \dfrac{p}{q}\,, \qquad p(0) = 8000\,, \\
 & \dot{q} = 5.85\, q^{2/3} - 0.00873\, q^{4/3} - 0.02\, q - 0.15\, q\, u\,, \\
 & \qquad\qquad q(0) = 10000\,, \\
 & \dot{y} = u\,, \qquad y(0) = 0\,, \\
 & y(t_f) \le 45\,, \quad 0 \le u(t) \le 15\,, \\
 & w\, p(t_f) \le \alpha\,, \\
 & (1 - w)\, (p(t_f) + 140\, y(t_f)) \le \alpha\,.
\end{array}
\right.
$$

Example 1 - Tumour Anti-angiogenesis

The switching function:

$$\sigma_u(t) = -0.15\,\lambda_q(t)\,q(t) + \lambda_y(t)\,.$$

Example 1 - Tumour Anti-angiogenesis

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Singular feedback control (Ledzewicz, Schättler, 2007, 2011):

$$u_{\text{sing}}(t) = \frac{1}{0.02}\left(\frac{5.85 - 0.00873\,q^{2/3}(t)}{q^{1/3}(t)} + 3\,(0.084)\,\frac{5.85 + 0.00873\,q^{2/3}(t)}{5.85 - 0.00873\,q^{2/3}(t)} - 0.02\right).$$

Example 1 - Tumour Anti-angiogenesis

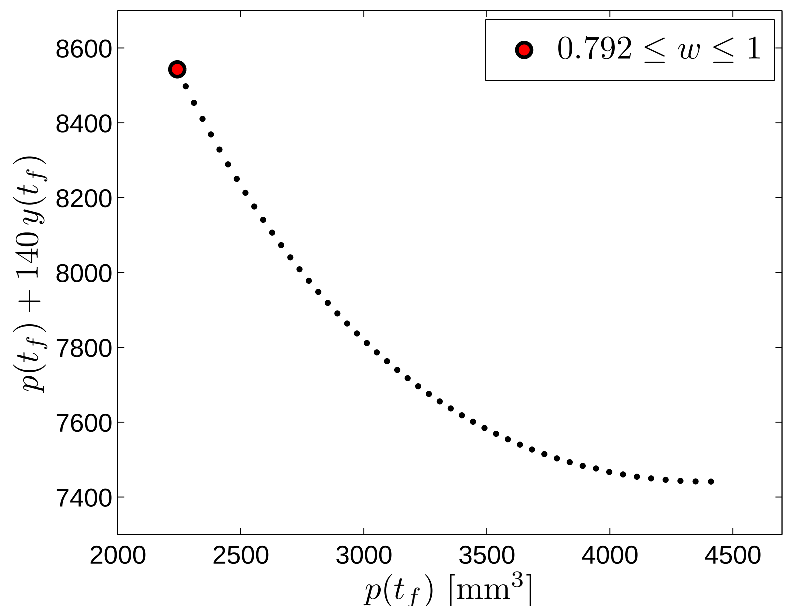
The optimal control turns out to be bang–singular–bang, in particular, to be full dose–partial dose–no dose.

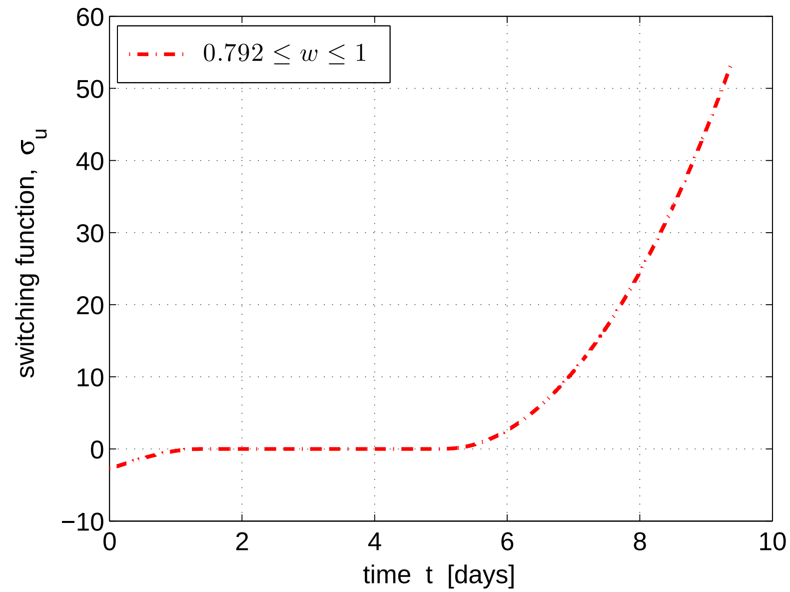
The switching times found accurately by means of *arc parameterization*: (Kaya & Noakes 2003, Maurer et al. 2005)

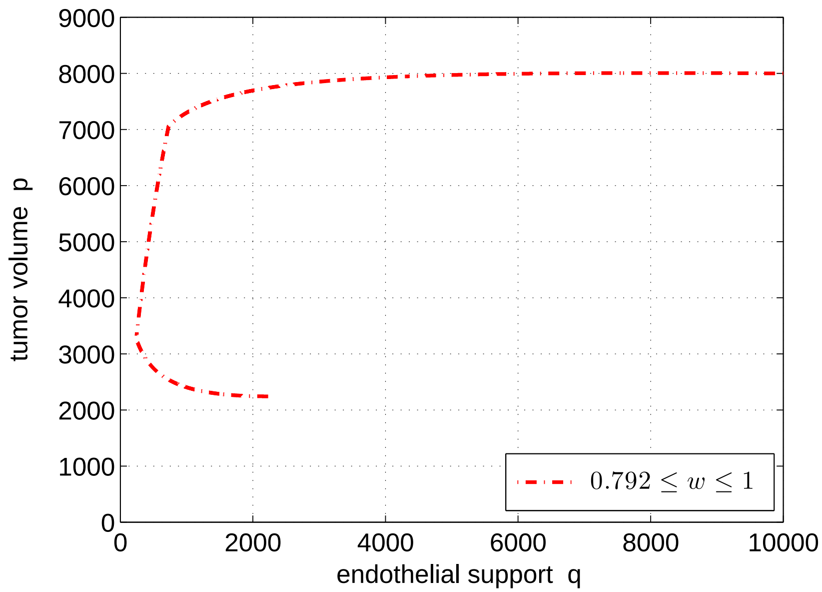
weights\switching times	t_1	t_2	t_f
$0.792 \le w \le 1$	1.341	5.063	9.378
$w = 0.7$	1.341	3.141	8.121
$0 \le w \le 0.628$	1.341	1.605	7.189

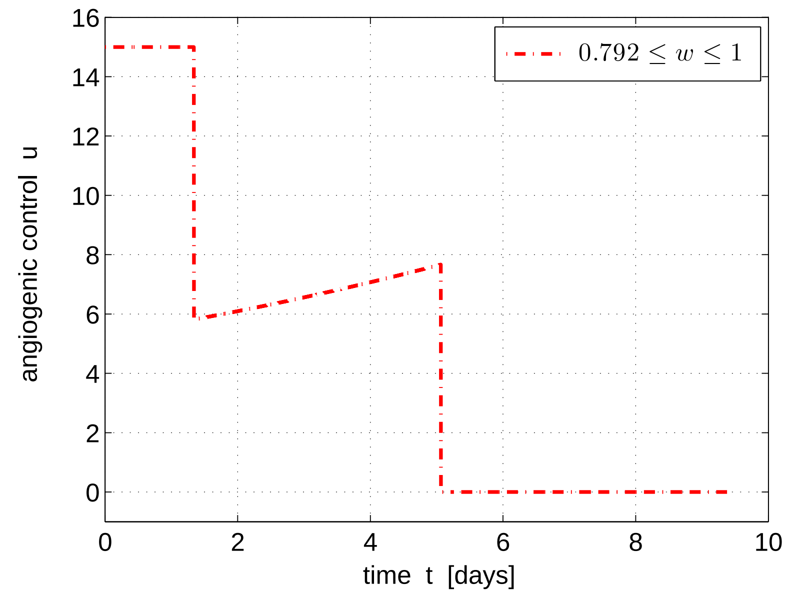
The meaningful interval is $w \in [\,0.628\,,\ 0.792\,]$.

Example 1 - Tumour Anti-angiogenesis









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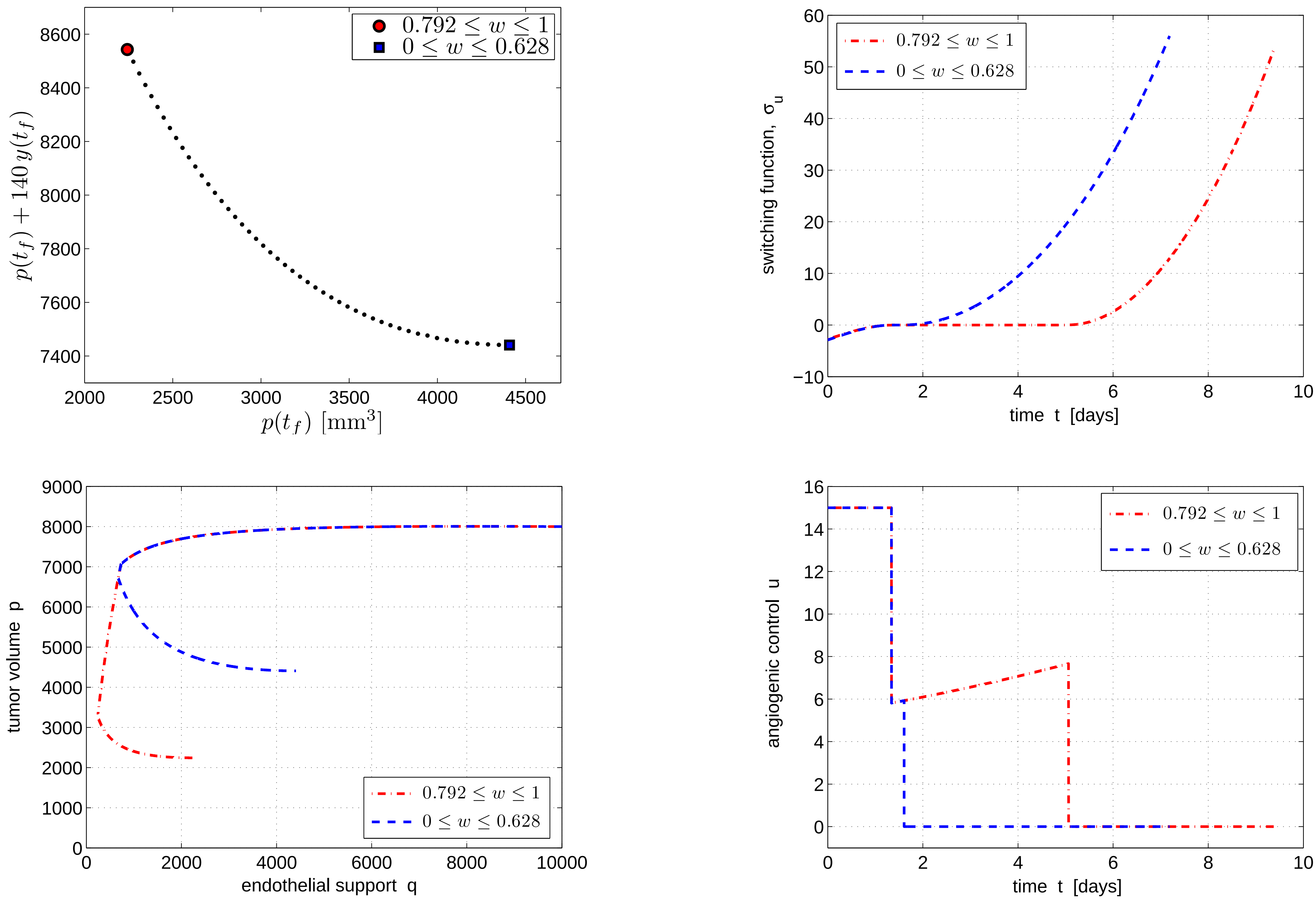

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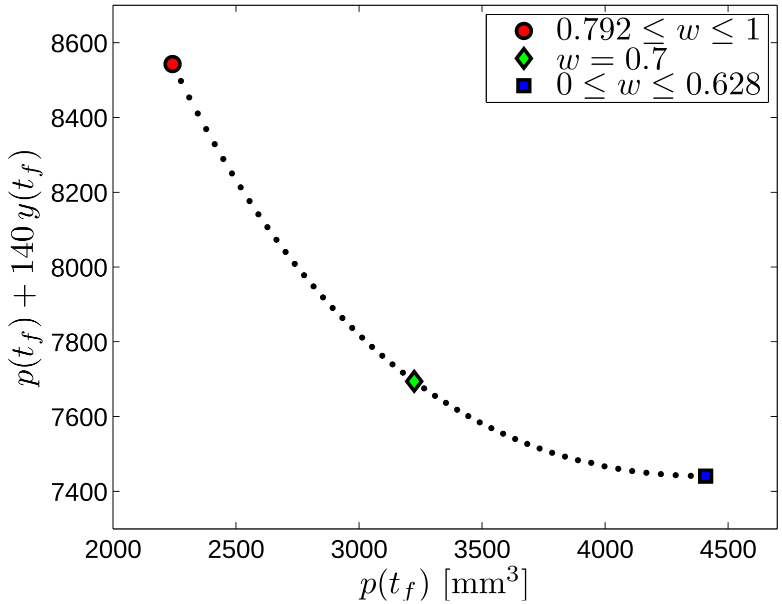

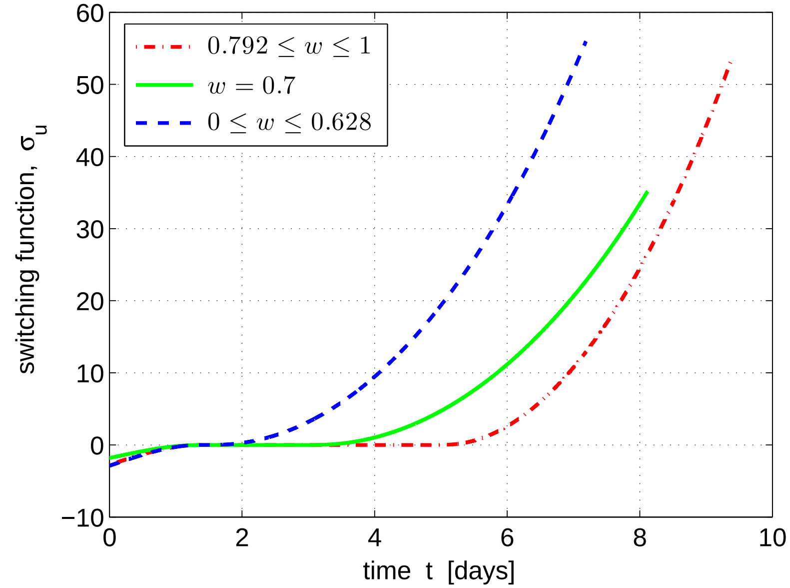

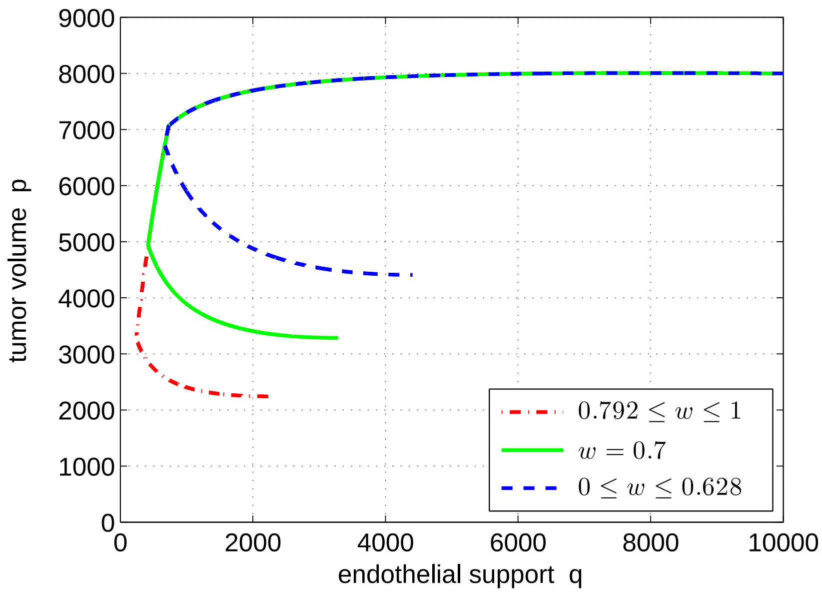

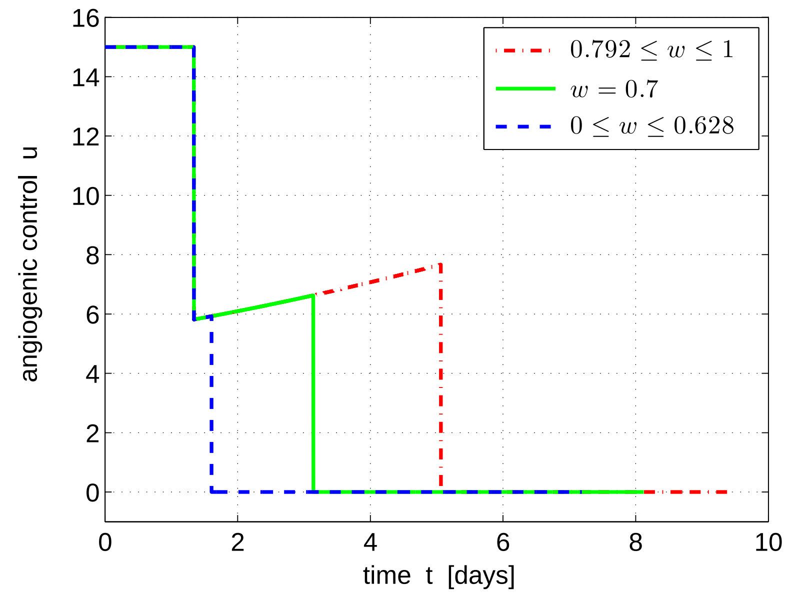

Example 2 - Fed-batch Bioreactor

(Logist, Houska, Diehl, van Impe, 2010)

The state and control variables:

x_1 : biomass [g]

x_2 : substrate [g]

x_3 : product, lysine [g]

x_4 : fermenter volume [g]

u : volumetric rate of the feed stream [lt/h]

Example 2 - Fed-batch Bioreactor

Two competing objective functionals to *maximize*:

- The ratio of the product formed and the process duration, i.e., the *productivity*:

$$\varphi_1(x(t_f), t_f) = \frac{x_3(t_f)}{t_f}.$$

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- The ratio of the product formed and the mass of the substrate added, i.e., the *yield*:

$$\varphi_2(x(t_f), t_f) = \frac{x_3(t_f)}{2.8\,(x_4(t_f) - 5)}.$$

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The duration of the process t_f is free.

Example 2 - Fed-batch Bioreactor

$$
\text{(E2)} \quad \begin{cases}
\min & \left(-\dfrac{x_3(t_f)}{t_f},\ -\dfrac{x_3(t_f)}{2.8\,(x_4(t_f)-5)} \right) \\
\text{s.t.} & \dot{x}_1 = \left(0.125\,\dfrac{x_2}{x_4} \right) x_1\,, \qquad x_1(0) = 0.1\,, \\
& \dot{x}_2 = -\left(\dfrac{0.125}{0.135}\,\dfrac{x_2}{x_4} \right) x_1 + 2.8\,u\,, \qquad x_2(0) = 14\,, \\
& \dot{x}_3 = -384\,\left(0.125\,\dfrac{x_2}{x_4} \right)^2 + 134\,\left(0.125\,\dfrac{x_2}{x_4} \right), \qquad x_3(0) = 0\,, \\
& \dot{x}_4 = u\,, \qquad x_4(0) = 5\,, \\
& 0 \le u(t) \le 2\,, \quad 5 \le x_4(t) \le 20 \\
& 0 \le t_f \le 40\,, \quad 20 \le 2.8\,(x_4(t_f) - 5) \le 42\,.
\end{cases}
$$

Expect bang-bang, singular and boundary arcs.

Example 2 - Fed-batch Bioreactor

Scalarization of Problem (E2): For a fixed $w \in [0,1]$, solve

$$(\mathrm{E2}_w)\ \left\{\begin{array}{rl} \min & \alpha \\ \text{s.t.} & \text{dynamical and terminal constraints,} \\ & \text{control and state constraints,} \\ & -w\left(\dfrac{x_3(t_f)}{t_f} + \beta_1^*\right) \le \alpha\,, \\ & -(1-w)\left(\dfrac{x_3(t_f)}{2.8\,(x_4(t_f)-5)} + \beta_2^*\right) \le \alpha\,, \end{array}\right.$$

where $\beta^* \le 0$ is a suitable utopia point.

Example 2 - Fed-batch Bioreactor

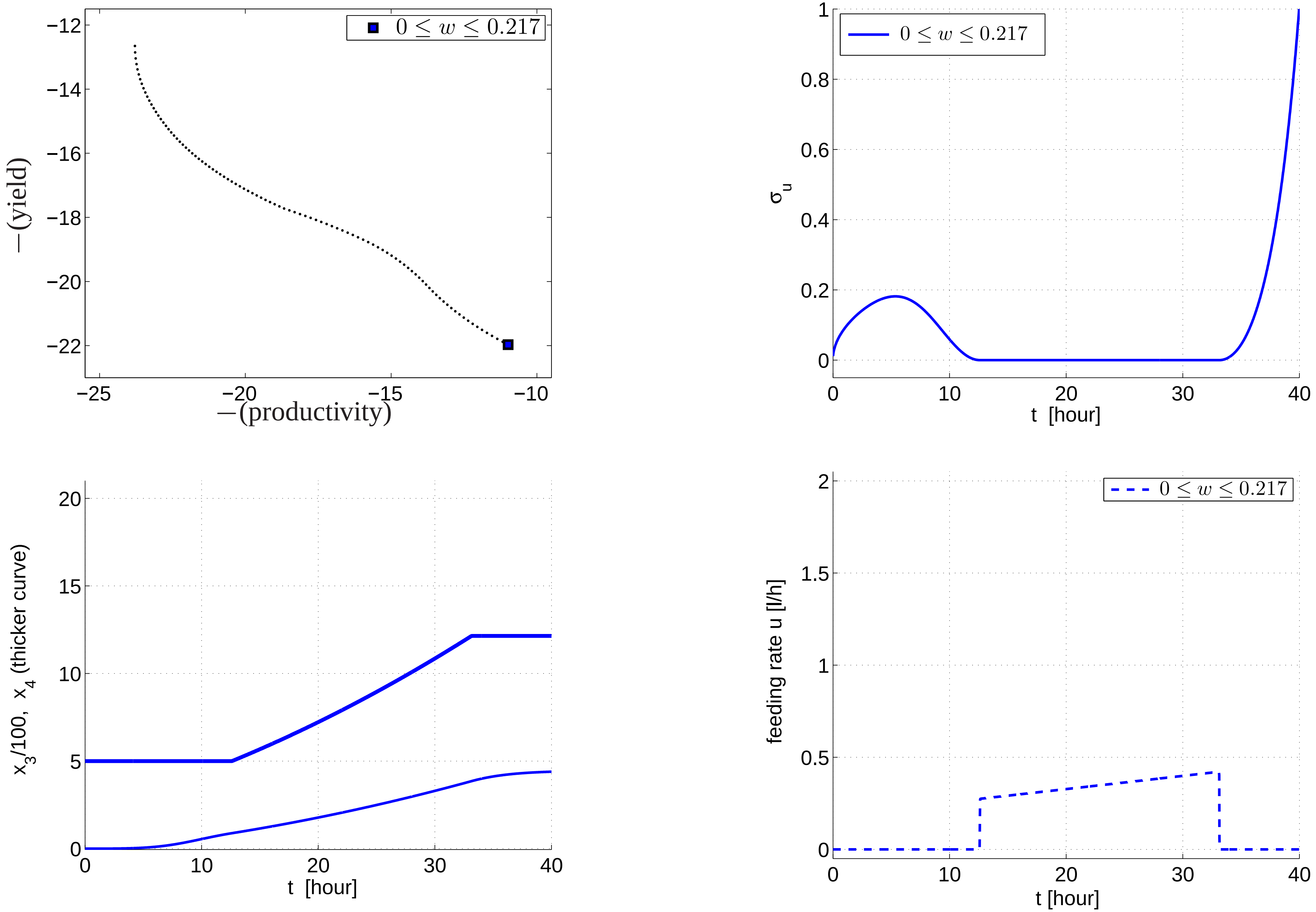
The state constraint $5 \leq x_4(t) \leq 20$ here is of order one.

The switching function is

$$\sigma_u(t) = 2.8\,\lambda_2(t)\,q(t) + \lambda_4(t)\,.$$

The singular arcs are of order one, but the singular control can not be obtained in feedback form.

Example 2 - Fed-batch Bioreactor



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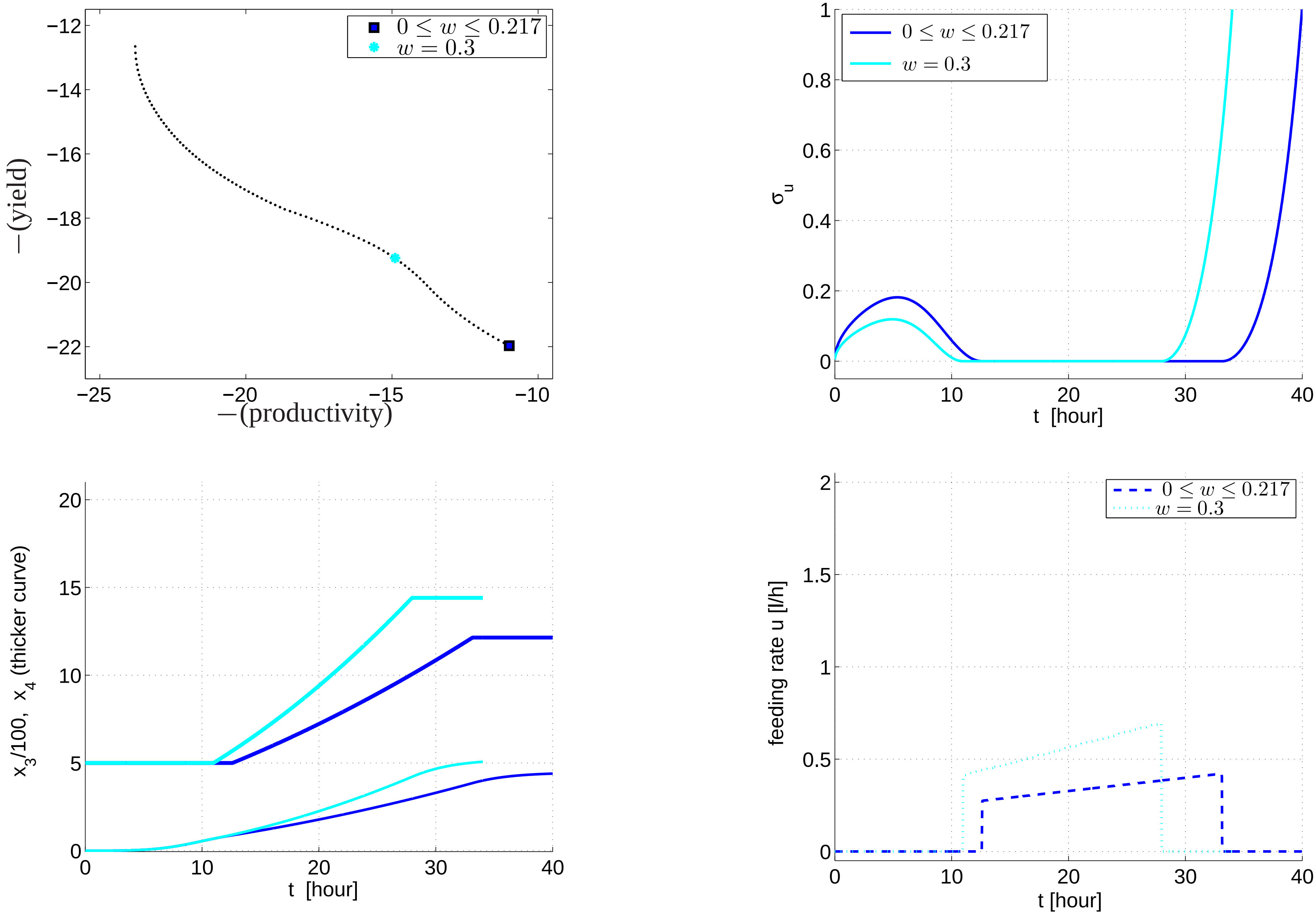

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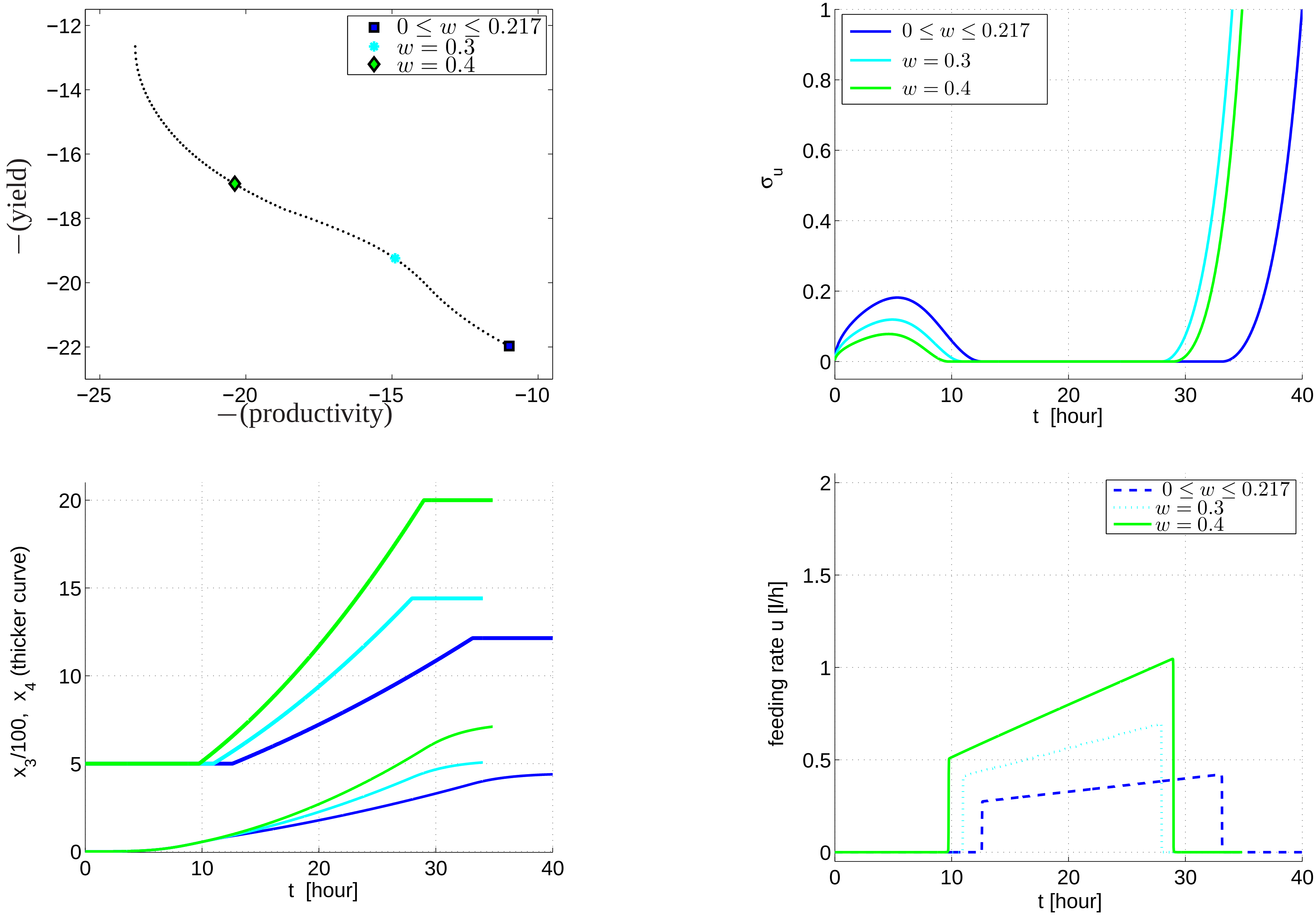

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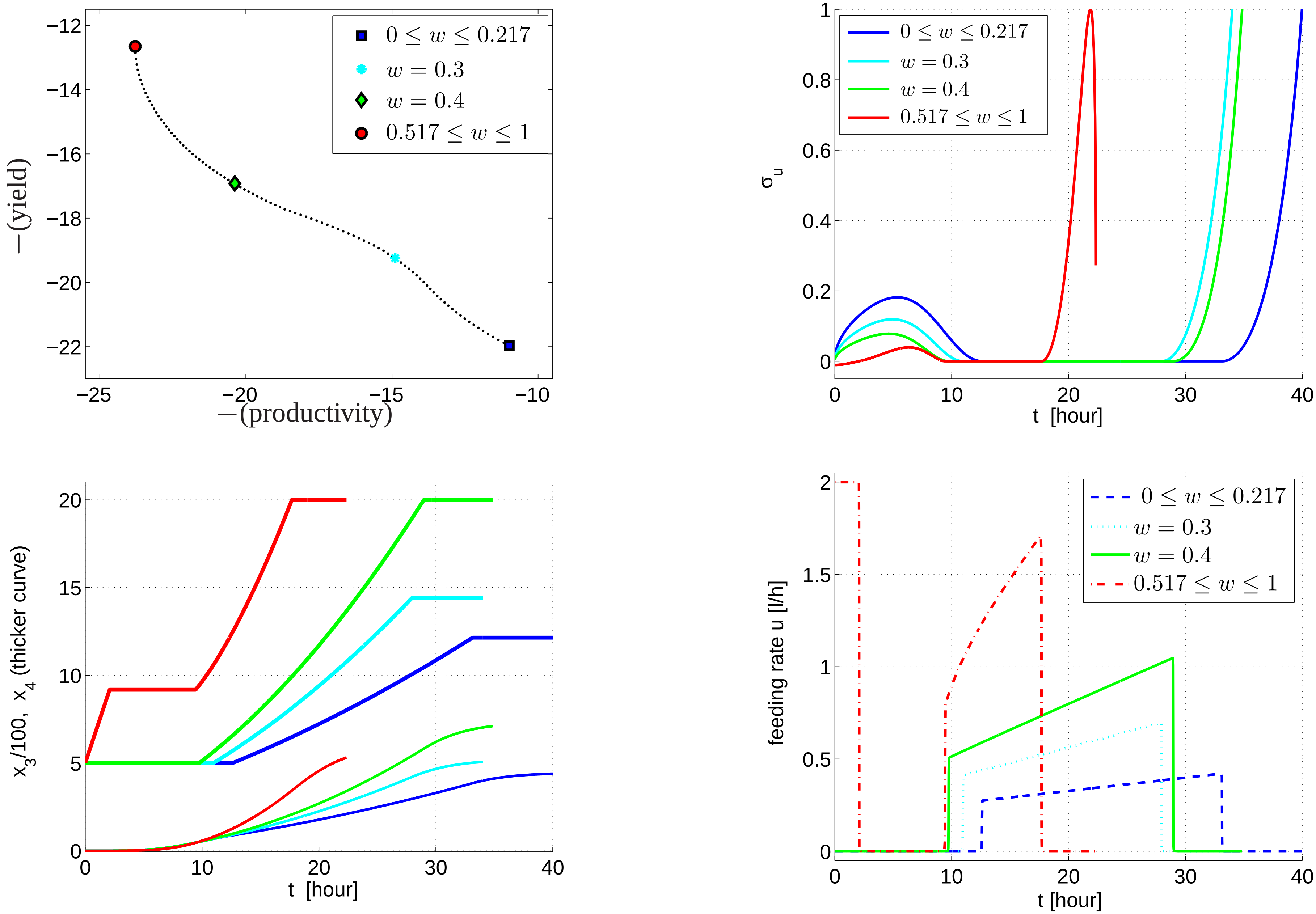
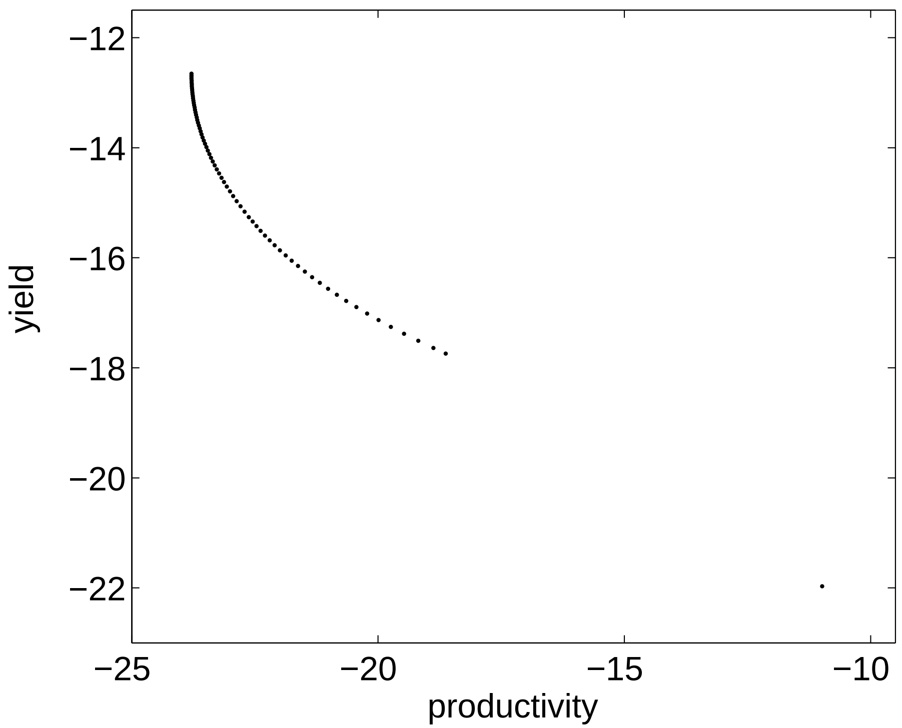


Example 2 - Fed-batch Bioreactor

Weighted-sum scalarization cannot generate "nonconvex parts" of the Pareto front:



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Future Work

1. Multi-objective control problems with $r \geq 3$ objectives.

2. Optimization over the Pareto front using sensitivity analysis.

3. Multi-objective control problems for elliptic and parabolic equations.

Conclusion

Use our method for multi-objective decisions !

Thank you for your attention !