Galerkin variational integrators in optimal control theory

Sina Ober-Blöbaum

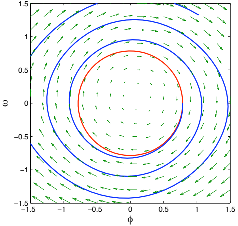
Computational Dynamics and Optimal Control
Department of Mathematics
University of Paderborn

AIMS 2014, July 7-11, 2014

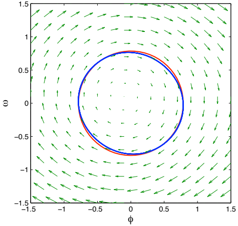
Joint work with Cédric M. Campos and Emmanuel Trélat

Motivation

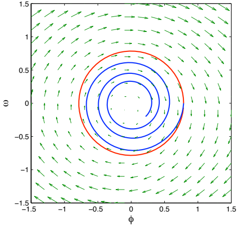
Variational (geometric) integration



explicit Euler



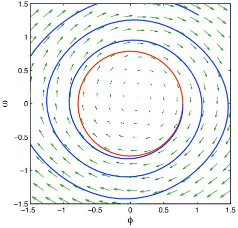
variational Euler



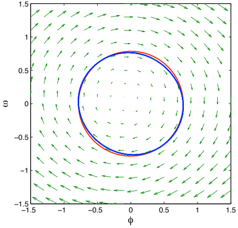
implicit Euler

Motivation

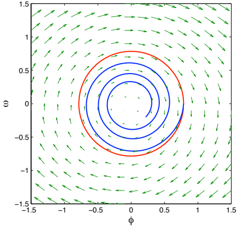
Variational (geometric) integration



explicit Euler



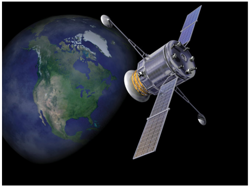
variational Euler



implicit Euler

Optimal control problem

- direct discretization approach via variational integration
- goal: preservation of geometric properties + high accuracy



Outline

- Problem formulation: optimal control for mechanical systems
- Galerkin variational integrators
  - construction, numerical analysis
- Galerkin variational integrators in optimal control
  - commutation of discretization and dualization
- Conclusion and future directions

Problem formulation

- configuration $q(t)$, velocity $\dot{q}(t)$, control $u(t)$
- Lagrangian $L(q(t), \dot{q}(t))$
- force $F(q(t), \dot{q}(t), u(t))$

Lagrangian optimal control problem (LOCP)

$$\min_{q,\dot{q},u} J(q, \dot{q}, u) = \int_0^T C(q(t), \dot{q}(t), u(t))\,\mathrm{d}t + \Phi(q(T), \dot{q}(T))$$

subject to the Lagrange-d'Alembert principle

$$\delta \int_0^{\mathrm{T}} L(q(t), \dot{q}(t))\,\mathrm{d}t + \int_0^{\mathrm{T}} F(q(t), \dot{q}(t), u(t)) \cdot \delta q(t)\; dt = 0$$

$$(q(0), \dot{q}(0)) = (q^0, \dot{q}^0)$$

Problem formulation

- configuration $q(t)$, velocity $\dot{q}(t)$, control $u(t)$
- Lagrangian $L(q(t), \dot{q}(t))$
- force $F(q(t), \dot{q}(t), u(t))$

Lagrangian optimal control problem (LOCP)

$$\min_{q,\dot{q},u} J(q, \dot{q}, u) = \int_0^T C(q(t), \dot{q}(t), u(t))\, \mathrm{d}t + \Phi(q(T), \dot{q}(T))$$

subject to the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}(q(t), \dot{q}(t)) - \frac{\partial L}{\partial q}(q(t), \dot{q}(t)) = F(q(t), \dot{q}(t), u(t))$$

$$(q(0), \dot{q}(0)) = (q^0, \dot{q}^0)$$

Problem formulation

- configuration $q(t)$, momentum $p(t)$, control $u(t)$
- Lagrangian $L(q(t), \dot{q}(t))$
- force $F(q(t), \dot{q}(t), u(t))$

Lagrangian optimal control problem (LOCP)

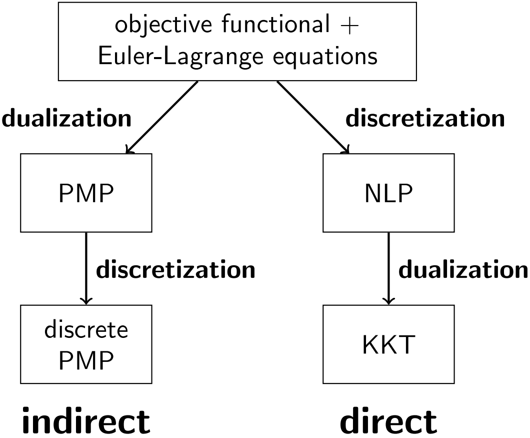
$$\min_{q,p,u} J(q,p,u) = \int_0^T C(q(t), p(t), u(t))\,\mathrm{d}t + \Phi(q(T), p(T))$$

subject to the Euler-Lagrange equations

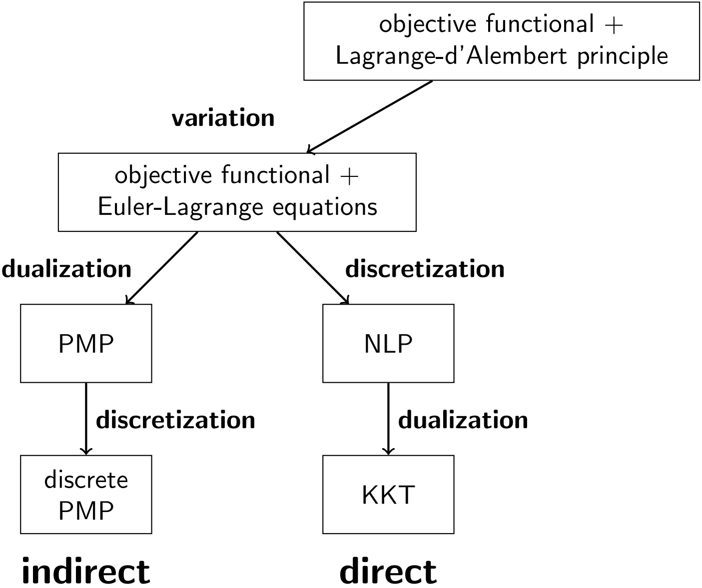
$$\begin{aligned} \dot{q}(t) &= f(q(t), p(t)), \qquad & q(0) &= q^0 \\ \dot{p}(t) &= g(q(t), p(t), u(t)), \qquad & p(0) &= p^0 \end{aligned}$$

with Legendre transform $p = \frac{\partial L}{\partial \dot{q}}(q, \dot{q})$ and $\dot{q} = \left(\frac{\partial L}{\partial \dot{q}}\right)^{-1}(q,p)$.

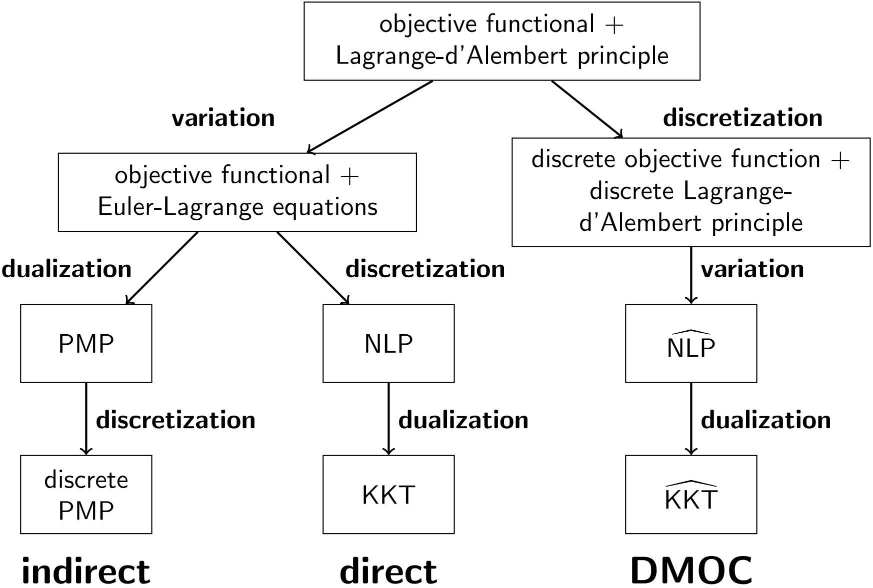
Numerical methods: overview



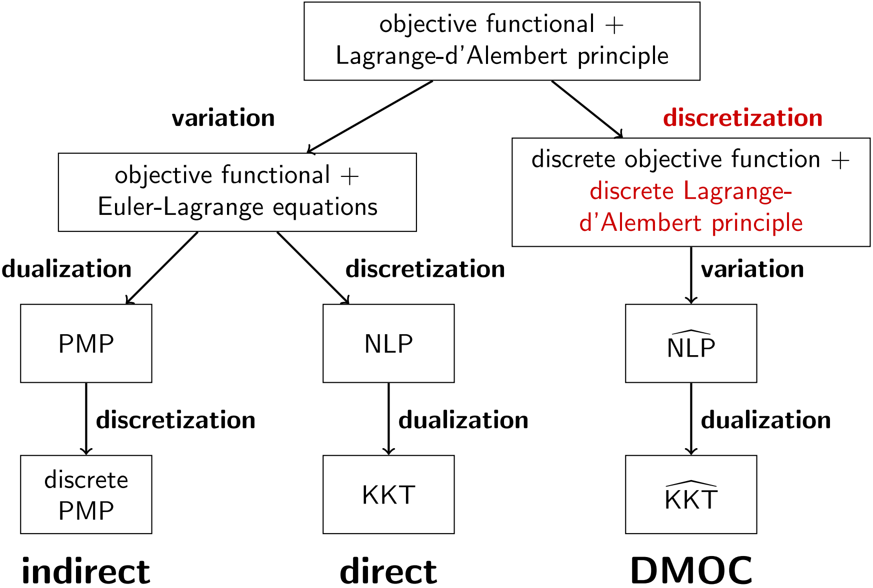
Numerical methods: overview



Numerical methods: overview



Numerical methods: overview



Lagrangian dynamics: variational principle

- Lagrangian $L : TQ \to \mathbb{R}$
- stationary action

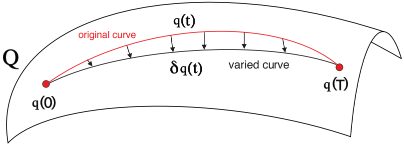
$$\delta \mathfrak{S} = \delta \int_0^T L(q, \dot{q})\, \mathrm{d}t = 0$$

- Euler-Lagange equations

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

- Lagrangian flow

$$F_L^t : TQ \to TQ$$



Lagrangian dynamics: variational principle

- Lagrangian $L : TQ \to \mathbb{R}$
- stationary action

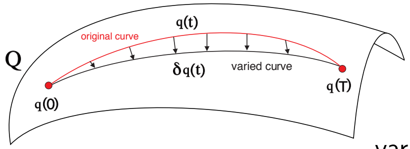
$$\delta\mathfrak{S} = \delta \int_0^T L(q, \dot{q})\, dt = 0$$

- Euler-Lagange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

- Lagrangian flow

$$F_L^t : TQ \to TQ$$



- discrete $L_d : Q \times Q \to \mathbb{R}$
- stationary discrete action

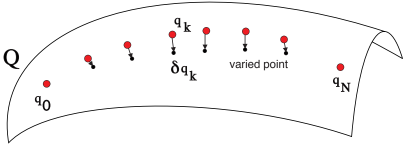
$$\delta\mathfrak{S}_d = \delta \sum_{k=0}^{N-1} L_d(q_k, q_{k+1}) = 0$$

- discrete Euler-Lagange eq.

$$D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) = 0$$

- discrete Lagrangian flow

$$F_{L_d}^h : Q \times Q \to Q \times Q$$



variational integrator: [Marsden, West 01]

Direct approach: DMOC

[O., Junge, Marsden 05]
[Leyendecker, O., Marsden, Ortiz 08]

Discrete Lagrangian optimal control problem

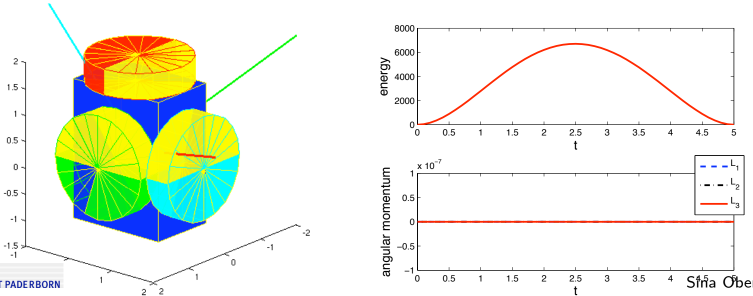
$$\min_{\{q_k,u_k\}_{k=0}^N} J_d(\{q_k, u_k\}_{k=0}^N)$$

subject to the discrete forced Euler-Lagrange equations

$$D_2L_d(q_{k-1}, q_k) + D_1L_d(q_k, q_{k+1}) + F_d^+(q_{k-1}, q_k) + F_d^-(q_k, q_{k+1}) = 0$$

$$q_0 = q^0,\ D_2L(q^0, \dot{q}^0) + D_1L_d(q_0, q_1) + F_0^- = 0$$

$k = 1, \dots, N-1.$



Direct approach: DMOC

[O., Junge, Marsden 05]
[Leyendecker, O., Marsden, Ortiz 08]

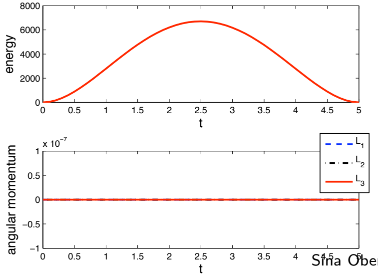
Discrete Lagrangian optimal control problem

$$\min_{\{q_k,u_k\}_{k=0}^N} J_d(\{q_k,u_k\}_{k=0}^N)$$

subject to the discrete forced Euler-Lagrange equations

$$\begin{aligned} D_2L_d(q_{k-1},q_k)+D_1L_d(q_k,q_{k+1})+F_d^+(q_{k-1},q_k)+F_d^-(q_k,q_{k+1}) &= 0 \\ q_0=q^0,\ D_2L(q^0,\dot{q}^0)+D_1L_d(q_0,q_1)+F_0^- &= 0 \end{aligned}$$

$k=1,\ldots,N-1.$



Galerkin variational integrators – construction

Approximation of action integral

$$\int_0^h L(q(t), \dot{q}(t))\, dt$$

1. approximation of the space of trajectories

$$\mathcal{C}([0,h], Q) = \{q : [0,h] \to Q \mid q(0) = q_0, q(h) = q_1\}$$

2. approximation of the integral of the Lagrangian by numerical quadrature rules

Galerkin variational integrators – construction

Approximation of action integral

$$\int_0^h L(q(t), \dot{q}(t))\, dt$$

1. approximation of the space of trajectories

$$\mathcal{C}([0,h],Q) = \{q : [0,h] \to Q \mid q(0) = q_0, q(h) = q_1\}$$

2. approximation of the integral of the Lagrangian by numerical quadrature rules

Related works:

- Galerkin variational integrators [Marsden, West (2001)]
- Galerkin and shooting based [Leok, Shingl (2010)]
- convergence analysis [Hall, Leok (2013)]

Galerkin variational integrators – construction

polynomial interpolation

- collocation points $0 \leq c_1 < \ldots < c_s \leq 1$
- Lagrange polynomials $l^j(t) := \prod_{i \neq j} \frac{t - c_i}{c_j - c_i}$

numerical quadrature

- quadrature points and weights $(c_i, b_i)_{i=1}^s$
- points based on Gauss, Lobatto, Radau, Chebyshev
- weights $b_j := \int_0^1 l^j(t) \mathrm{d}t$

two variational integrators

- based on different dimensions of approximation space
- symplectic partitioned Runge-Kutta + symplectic Galerkin

[Campos: High order variational integrators: a polynomial approach. In *23rd Congress on Differential Equations and Applications*, Springer Series 2014]

[Campos, O., Trélat: High order variational integrators in the optimal control of mechanical systems, Preprint 2014]

Galerkin variational integrators – construction

symplectic partitioned Runge-Kutta	symplectic Galerkin
micro velocities $\dot{Q}_1, \ldots, \dot{Q}_s$	
interpolation polynomial $\dot{Q}(t) = \sum_{j=1}^{s} l^j(t/h)\dot{Q}_j$ $Q(t) = q_0 + h\sum_{j=1}^{s} \int_0^{t/h} l^j(\tau)\mathrm{d}\tau\,\dot{Q}_j$	
internal stage constraint $Q_i = Q(h \cdot c_i) = q_0 + h\sum_{j=1}^{s} a_{ij}\dot{Q}_j$ with $a_{ij} := \int_0^{c_i} l^j(t)\mathrm{d}t$	
boundary nodes $q_0 = Q(0)$ $q_1 = Q(h) = q_0 + h\sum_{j=1}^{s} b_j\dot{Q}_j$	

Galerkin variational integrators – construction

symplectic partitioned Runge-Kutta	symplectic Galerkin
micro velocities $\dot{Q}_1, \ldots, \dot{Q}_s$	micro nodes $Q_1, \ldots, Q_s$
interpolation polynomial	
$\dot{Q}(t) = \sum_{j=1}^{s} l^j(t/h)\dot{Q}_j$	$Q(t) = \sum_{j=1}^{s} l^j(t/h) Q_j$
$Q(t) = q_0 + h \sum_{j=1}^{s} \int_0^{t/h} l^j(\tau) \mathrm{d}\tau \, \dot{Q}_j$	$\dot{Q}(t) = \frac{1}{h} \sum_{j=1}^{s} \dot{l}^j(t/h) Q_j$
internal stage constraint	
$Q_i = Q(h \cdot c_i) = q_0 + h \sum_{j=1}^{s} a_{ij} \dot{Q}_j$	$\dot{Q}_i = \dot{Q}(h \cdot c_i) = \frac{1}{h} \sum_{j=1}^{s} a_{ij} Q_j$
with $a_{ij} := \int_0^{c_i} l^j(t) \mathrm{d}t$	with $a_{ij} = \left.\frac{\mathrm{d}l^j}{\mathrm{d}t}\right\vert_{c_i}$
boundary nodes	
$q_0 = Q(0)$	$q_0 = Q(0) = \sum_{j=1}^{s} \alpha^j Q_j$
$q_1 = Q(h) = q_0 + h \sum_{j=1}^{s} b_j \dot{Q}_j$	$q_1 = Q(h) = \sum_{j=1}^{s} \beta^j Q_j$
	with $\alpha^j := l^j(0), \; \beta^j := l^j(1)$

Galerkin variational integrators – construction

symplectic partitioned Runge-Kutta	symplectic Galerkin

multi-point discrete Lagrangian

$$L_d(\dot{Q}_1,\ldots,\dot{Q}_s) = h\sum_{i=1}^s b_i L(Q_i,\dot{Q}_i)$$

two-point discrete Lagrangian

$$L_d(q_0,q_1) = \underset{\mathcal{P}^s([0,h],\mathbb{R}^n,q_0,q_1)}{\mathrm{ext}} L_d(\dot{Q}_1,\ldots,\dot{Q}_s)$$

symplectic partitioned RK scheme

$$q_1 = q_0 + h\sum_{i=1}^s b_i\dot{Q}_i,\ p_1 = p_0 + h\sum_{i=1}^s \bar{b}_i\dot{P}_i$$

$$Q_i = q_0 + h\sum_{j=1}^s a_{ij}\dot{Q}_j,\ P_i = p_0 + h\sum_{j=1}^s \bar{a}_{ij}\dot{P}_j,$$

$$P_i = \frac{\partial L}{\partial \dot{q}}(Q_i,\dot{Q}_i),\ \dot{P}_i = \frac{\partial L}{\partial q}(Q_i,\dot{Q}_i)$$

$$i = 1,\ldots,s$$

$$\bar{a}_{ij} = b_j - b_j a_{ji}/b_i,\ b_i = \bar{b}_i$$

[Marsden, West 2001]
[Hairer, Wanner, Lubich 2000]

Galerkin variational integrators – construction

symplectic partitioned Runge-Kutta	symplectic Galerkin
multi-point discrete Lagrangian	
$L_d(\dot{Q}_1, \dots, \dot{Q}_s) = h \sum_{i=1}^s b_i L(Q_i, \dot{Q}_i)$	$L_d(Q_1, \dots, Q_s) = h \sum_{i=1}^s b_i L(Q_i, \dot{Q}_i)$
two-point discrete Lagrangian	
$L_d(q_0, q_1) = \underset{\mathcal{P}^s([0,h],\mathbb{R}^n,q_0,q_1)}{\text{ext}} L_d(\dot{Q}_1, \dots, \dot{Q}_s)$	$L_d(q_0, q_1) = \underset{\mathcal{P}^{s-1}([0,h],\mathbb{R}^n,q_0,q_1)}{\text{ext}} L_d(Q_1, \dots, Q_s)$
symplectic partitioned RK scheme	**symplectic Galerkin scheme**
$q_1 = q_0 + h \sum_{i=1}^s b_i \dot{Q}_i,\ p_1 = p_0 + h \sum_{i=1}^s \bar{b}_i \dot{P}_i$	$q_0 = \sum_{j=1}^s \alpha^j Q_j,\ q_1 = \sum_{j=1}^s \beta^j Q_j$
$Q_i = q_0 + h \sum_{j=1}^s a_{ij} \dot{Q}_j,\ P_i = p_0 + h \sum_{j=1}^s \bar{a}_{ij} \dot{P}_j,$	$\dot{Q}_i = \frac{1}{h} \sum_{j=1}^s a_{ij} Q_j,\ \dot{P}_i = \frac{\beta^i p_1 - \alpha^i p_0}{h \bar{b}_i} + \frac{1}{h} \sum_{j=1}^s \bar{a}_{ij} P_j$
$P_i = \frac{\partial L}{\partial \dot{q}}(Q_i, \dot{Q}_i),\ \dot{P}_i = \frac{\partial L}{\partial q}(Q_i, \dot{Q}_i)$	$P_i = \frac{\partial L}{\partial \dot{q}}(Q_i, \dot{Q}_i),\ \dot{P}_i = \frac{\partial L}{\partial q}(Q_i, \dot{Q}_i)$
$i = 1, \dots, s$	$i = 1, \dots, s$
$\bar{a}_{ij} = b_j - b_j a_{ji}/b_i,\ b_i = \bar{b}_i$	$b_i a_{ij} + \bar{b}_j \bar{a}_{ji} = 0,\ b_i = \bar{b}_i$
[Marsden, West 2001] [Hairer, Wanner, Lubich 2000]	[Campos 2014] [Campos, O., Trélat 2014]

Equivalence of spRK and sG

- sRK: $Q \in \mathcal{P}^s, \ \dot{Q} \in \mathcal{P}^{s-1}$
- sG: $Q \in \mathcal{P}^{s-1}, \ \dot{Q} \in \mathcal{P}^{s-2}$
- s internal stages Q_i and $\dot{Q}_i$

$$\Rightarrow \exists l : d_l \neq 0, \sum_{i=1}^{s} d_i \dot{Q}_i = 0$$

- consider $\sum_{i=1}^{s} d_i \dot{Q}_i = 0$ as additional constraint for extremizing L_d

equivalent to a "spRK" of the form

$$\begin{aligned}
q_1 &= q_0 + h \sum_{i=1}^{s} b_i \dot{Q}_i, & p_1 &= p_0 + h \sum_{i=1}^{s} \bar{b}_i \dot{P}_i, \\
Q_i &= q_0 + h \sum_{j=1}^{s} a_{ij} \dot{Q}_j, & P_i &= p_0 + h \sum_{j=1}^{s} \bar{a}_{ij} \dot{P}_j - \frac{d_i}{b_i} \lambda, \\
P_i &= \frac{\partial L}{\partial \dot{q}}(Q_i, \dot{Q}_i) & \dot{P}_i &= \frac{\partial L}{\partial q}(Q_i, \dot{Q}_i) \\
0 &= \textstyle\sum_{i=1}^{s} d_i \dot{Q}_i
\end{aligned}$$

Relation to Runge-Kutta schemes

Theorem (Equivalence to Runge-Kutta scheme)

For a Lagrangian of the form $L = \frac{1}{2}\dot{q}^T M \dot{q} - V(q)$, *the sG variational integrator is equivalent to an s-stage symplectic partitioned Runge-Kutta scheme if the following conditions on the coefficients hold*

$$\sum_{i=1}^{s} d_i \bar{a}_{ij} = 0,\ j = 1, \ldots, s, \quad \sum_{i=1}^{s} d_i = 0, \quad \sum_{i=1}^{s} \frac{d_i^2}{b_i} \neq 0$$

Remark:

- satisfied for all schemes based on the Lobatto quadrature

[O.: Galerkin variational integrators and modified symplectic Runge-Kutta methods, Preprint 2014]

Galerkin variational integrators – order

		spRK	sG
micro-data		$\dot{Q}_i$	Q_i
polynomial degree		s	$s-1$
quadrature	Gauss	$2s$	$2s-2$
	Lobatto	$2s-2$	$2s-2$
	Radau	$2s-1$	$2s-2$
	Chebyshev	$2s-2$	$2s-2$
		order method	

Table: Comparison of s-stage variational integrators.

examples

- spRK with Gauss: Gauss collocation method
- spRK with Lobatto: Lobatto IIIA-IIIB partitioned Runge-Kutta method

[Marsden, West 2001], [Hairer, Wanner, Lubich 2000]

Optimal control problem

Lagrangian optimal control problem (LOCP)

$$\min_{q,p,u} J(q,p,u) = \int_0^T C(q(t),p(t),u(t))\,\mathrm{d}t + \Phi(q(T),p(T))$$

subject to the Euler-Lagrange equations

$$\begin{aligned} \dot{q}(t) &= f(q(t),p(t)), \qquad & q(0) &= q^0 \\ \dot{p}(t) &= g(q(t),p(t),u(t)), \qquad & p(0) &= p^0 \end{aligned}$$

with Legendre transform $p = \frac{\partial L}{\partial \dot{q}}(q,\dot{q})$ and $\dot{q} = \left(\frac{\partial L}{\partial \dot{q}}\right)^{-1}(q,p)$.

Discretized optimal control problem

Discrete optimal control problem: the spRK case

$$\min_{q_d, p_d, \{Q_i^k, P_i^k, U_i^k\}_{i=1,\ldots,s}^{k=0,\ldots,N-1}} J_d(q_d, p_d, \{Q_i^k, P_i^k, U_i^k\}_{i=1,\ldots,s}^{k=0,\ldots,N-1})$$

subject to

$$q_{k+1} = q_k + h\sum_{j=1}^{s} b_j f(Q_j^k, P_j^k), \; p_{k+1} = p_k + h\sum_{j=1}^{s} \bar{b}_j g(Q_j^k, P_j^k, U_j^k)$$

$$Q_i^k = q_k + h\sum_{j=1}^{s} a_{ij} f(Q_j^k, P_j^k), \; P_i^k = p_k + h\sum_{j=1}^{s} \bar{a}_{ij} g(Q_j^k, P_j^k, U_j^k)$$

$k = 0, \ldots, N-1$, $i = 1, \ldots, s$, with $b_i \bar{a}_{ij} + \bar{b}_j a_{ji} = b_i \bar{b}_j$ and $b_i = \bar{b}_i$,

$$(q_0, p_0) = (q^0, p^0).$$

Discretized optimal control problem

Discrete optimal control problem: the sG case

$$\min_{q_d, p_d, \{Q_i^k, P_i^k, U_i^k\}_{i=1,\dots,s}^{k=0,\dots,N-1}} J_d(q_d, p_d, \{Q_i^k, P_i^k, U_i^k\}_{i=1,\dots,s}^{k=0,\dots,N-1})$$

subject to

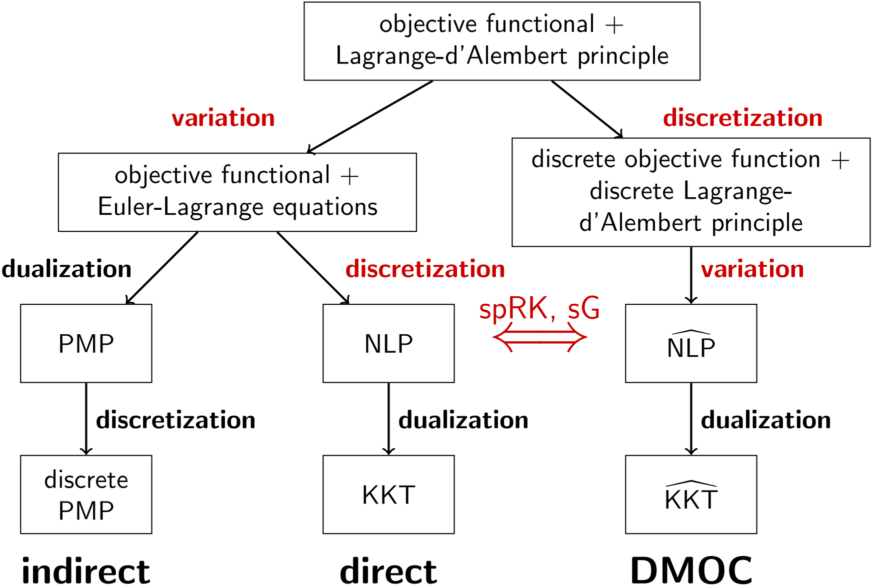
$$q_k = \sum_{j=1}^{s} \alpha^j Q_j^k, \; q_{k+1} = \sum_{j=1}^{s} \beta^j Q_j^k$$

$$f(Q_i^k, P_i^k) = \frac{1}{h} \sum_{j=1}^{s} a_{ij} Q_j^k, \; g(Q_i^k, P_i^k, U_i^k) = \frac{\beta^i p_{k+1} - \alpha^i p_k}{h \bar{b}_i} + \frac{1}{h} \sum_{j=1}^{s} \bar{a}_{ij} P_j^k$$

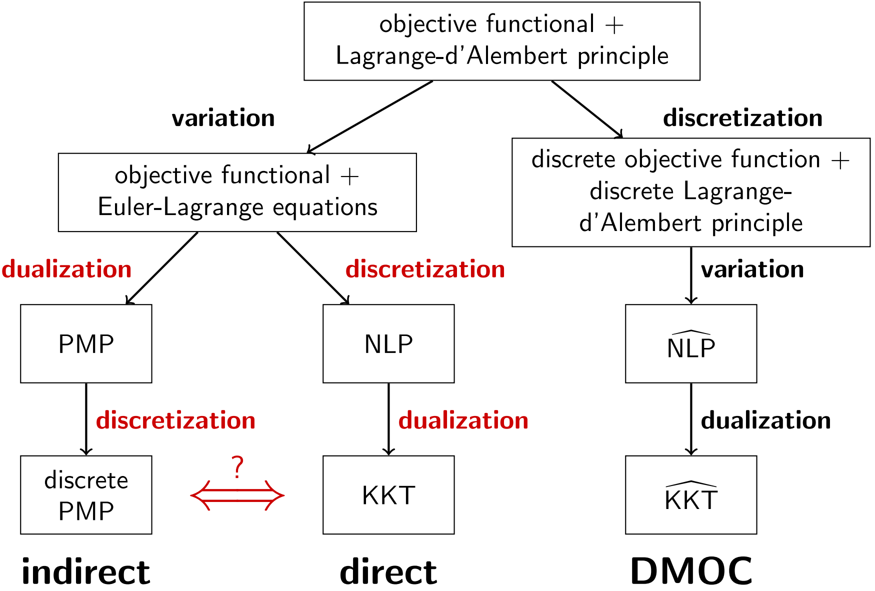
$k = 0, \dots, N-1$, $i = 1, \dots, s$, with $b_i a_{ij} + \bar{b}_j \bar{a}_{ji} = 0$ and $b_i = \bar{b}_i$,

$$(q_0, p_0) = (q^0, p^0).$$

Commutation property: discretization – variation



Commutation property: discretization – dualization



Commutation property: discretization – dualization

some known results

- RK methods:
  - · no commutation: different state and adjoint schemes
    $\Rightarrow$ order loss possible
  - · order conditions on Runge-Kutta coefficients to ensure order preservation [Hager 2000], [Bonnans at al. 2006]
- pseudospectral methods:
  - · commutation if extra closure conditions are satisfied
  - · order-preserving map between adjoint variables (covector mapping principle)
    [Gong, Ross, Kang, Fahroo 2008], [Ross, Fahroo 2003]
- symplectic partitioned Runge-Kutta methods:
  - · commutation due to symplecticity
  - · adjoint scheme is also symplectic

[O. 2008], [Sanz-Serna, Preprint 2014]

Commutation property: the sG case

- adjoint vectors $\lambda_0, \ldots, \lambda_N$, $\mu_0, \ldots, \mu_{N-1}$, ψ_0, $\Lambda_i^0, \ldots, \Lambda_i^{N-1}$, $\Psi_i^0, \ldots, \Psi_i^{N-1}$, $i = 1, \ldots, s$
- discrete optimal control Lagrangian

$$\begin{aligned}\mathcal{L}_d \quad = \quad & \Phi(q_N, p_N) - \lambda_0^T(q_0 - q^0) - \psi_0^T(p_0 - p^0) \\ & + \sum_{k=0}^{N-1}\Bigg[\mu_k^T\left(q_k - \sum_{j=1}^{s}\alpha^j Q_j^k\right) - \lambda_{k+1}^T\left(q_{k+1} - \sum_{j=1}^{s}\beta^j Q_j^k\right) \\ & + \sum_{i=1}^{s}\left(\Lambda_i^k\right)^T\left(hf_i^k - \sum_{j=1}^{s}a_{ij}Q_j^k\right) + \left(\Psi_i^k\right)^T\left(hg_i^k - \frac{\beta^i p_{k+1} - \alpha^i p_k}{\bar{b}_i} - \sum_{j=1}^{s}\bar{a}_{ij}P_j^k\right)\Bigg]\end{aligned}$$

with $f_i^k = f(Q_i^k, P_i^k)$ and $g_i^k = g(Q_i^k, P_i^k, U_i^k)$.

Commutation property: the sG case

- new adjoint variables
  $\Gamma_i^k := \frac{\Lambda_i^k}{\bar{b}_i}$ and $\chi_i^k := \frac{\Psi_i^k}{\bar{b}_i}$, $k = 0, \dots, N-1$, $i = 1, \dots, s$,
- $\psi_k^- := \sum_{i=1}^s \chi_i^k \alpha^i, \quad \psi_k^+ := \sum_{i=1}^s \chi_i^{k-1} \beta^i$
- KKT equation

$$
\begin{aligned}
\psi_k &= \sum_{j=1}^{s} \alpha^j \chi_j^k, \\
\psi_{k+1} &= \sum_{j=1}^{s} \beta^j \chi_j^k, \\
-f_q^T(Q_i^k, P_i^k)\,\Gamma_i^k - g_q^T(Q_i^k, P_i^k, U_i^k)\,\chi_i^k &= \frac{\beta^i \lambda_{k+1} - \alpha^i \lambda_k}{h\bar{b}_i} + \frac{1}{h}\sum_{j=1}^{s} \bar{a}_{ij}\Gamma_j^k, \\
-f_p^T(Q_i^k, P_i^k)\,\Gamma_i^k - g_p^T(Q_i^k, P_i^k, U_i^k)\,\chi_i^k &= \frac{1}{h}\sum_{j=1}^{s} a_{ij}\chi_j^k, \\
g_u^T(Q_i^k, P_i^k, U_i^k)\,\Psi_i^k &= 0,
\end{aligned}
$$

for $k = 0, \dots, N-1$, $i = 1, \dots, s$, $b_i a_{ij} + \bar{b}_j \bar{a}_{ji} = 0$ and $b_i = \bar{b}_i$

$$\lambda_N = \Phi_q^T(q_N, p_N) \quad \text{and} \quad \psi_N = \Phi_p^T(q_N, p_N),$$

Commutation property: the sG case

- assumption: unique control

$$\begin{aligned}\nu(q,p,\lambda,\psi) &= \left(-f_q^T(q,p)\,\lambda - g_q^T(q,p,u)\,\psi\right)\big|_{u=u(q,p,\lambda,\psi)}\\ \eta(q,p,\lambda,\psi) &= \left(-f_p^T(q,p)\,\lambda - g_p^T(q,p,u)\,\psi\right)\big|_{u=u(q,p,\lambda,\psi)}.\end{aligned}$$

- necessary optimality conditions

state system

$$\begin{aligned}\dot q(t) &= f(q(t),p(t)), \quad q(0)=q^0\\ \dot p(t) &= g(q(t),p(t),\lambda(t),\psi(t)), \quad p(0)=p^0\end{aligned}$$

adjoint system

$$\begin{aligned}\dot\lambda(t) &= \nu(q(t),p(t),\lambda(t),\psi(t)), \quad \lambda(T)=\Phi_q(q(T),p(T))\\ \dot\psi(t) &= \eta(q(t),p(t),\lambda(t),\psi(t)), \quad \psi(T)=\Phi_q(q(T),p(T))\end{aligned}$$

Commutation property: the sG case

- assumption: unique control

$$\begin{aligned} \nu(q,p,\lambda,\psi) &= \left(-f_q^T(q,p)\,\lambda - g_q^T(q,p,u)\,\psi\right)\big|_{u=u(q,p,\lambda,\psi)} \\ \eta(q,p,\lambda,\psi) &= \left(-f_p^T(q,p)\,\lambda - g_p^T(q,p,u)\,\psi\right)\big|_{u=u(q,p,\lambda,\psi)}. \end{aligned}$$

- discrete necessary optimality conditions

$$\begin{aligned} q_k &= \sum_{j=1}^{s} \alpha^j Q_j^k\,, \quad q_{k+1} = \sum_{j=1}^{s} \beta^j Q_j^k\,, \quad \psi_k = \sum_{j=1}^{s} \alpha^j \chi_j^k\,, \quad \psi_{k+1} = \sum_{j=1}^{s} \beta^j \chi_j^k \\ f_i^k &= \frac{1}{h}\sum_{j=1}^{s} a_{ij} Q_j^k\,, \quad \eta_i^k = \frac{1}{h}\sum_{j=1}^{s} a_{ij}\chi_j^k \\ g_i^k &= \frac{\beta^i p_{k+1} - \alpha^i p_k}{h\bar{b}_i} + \frac{1}{h}\sum_{j=1}^{s} \bar{a}_{ij} P_j^k\,, \quad \nu_i^k = \frac{\beta^i \lambda_{k+1} - \alpha^i \lambda_k}{h\bar{b}_i} + \frac{1}{h}\sum_{j=1}^{s} \bar{a}_{ij} \Gamma_j^k\,, \end{aligned}$$

$k = 0, \dots, N-1$, $i = 1, \dots, s$, $\quad b_i a_{ij} + \bar{b}_j \bar{a}_{ji} = 0$ and $b_i = \bar{b}_i$

$$q_0 = q^0, \quad p_0 = p^0, \quad \lambda_N = \Phi_q(q_N, p_N), \quad \psi_N = \Phi_p(q_N, p_N)$$

Commutation property: discretization – dualization

Theorem (Commutation property)

Let a Lagrangian optimal control problem be given. If a symplectic Galerkin method with $b_i > 0$, $i = 1, \ldots, s$, is used for the discretization of the state system, dualization and discretization commute.

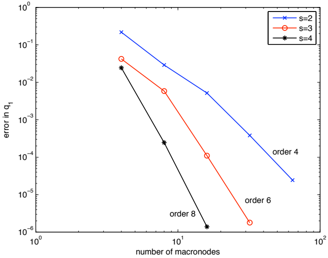
- order preservation
- scheme preservation (symplectic state and adjoint scheme)

$\Rightarrow$ convergence of state and adjoint variables under appropriate assumptions

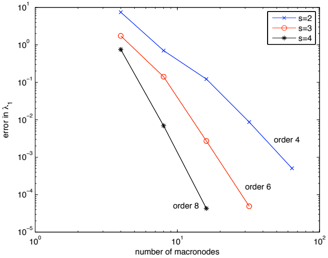
[Campos, O., Trélat: High order variational integrators in the optimal control of mechanical systems, Preprint 2014]

Convergence rates: states and adjoints

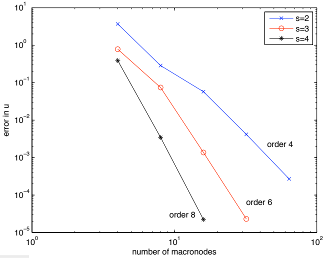
state error



adjoint error



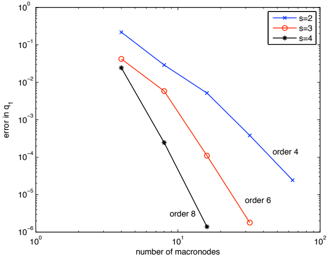
control error



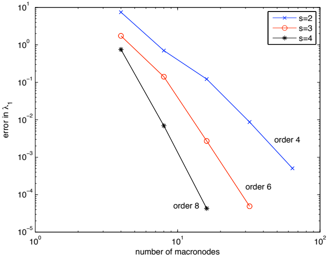
- orbital transfer with minimal control effort
- same convergence rates for configuration, control and adjoint

Convergence rates: states and adjoints

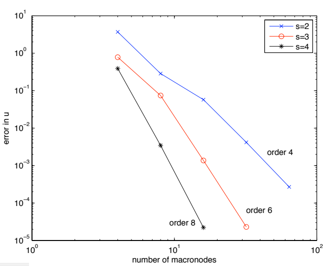
state error



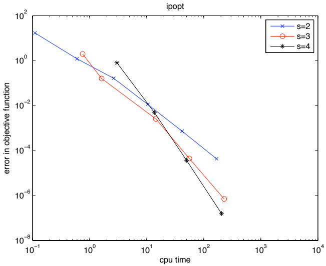
adjoint error



control error



error vs. cpu



Conclusion and future directions

Galerkin variational integrators

- symplectic-momentum preserving discretization
- based on polynomial interpolation and quadrature rules
- commutation of discretization and variation: spRK and sG
- equivalence of spRK and sG for Lobatto quadrature and separable L

Numerical optimal control

- commutation of discretization and dualization for spRK and sG

Conclusion and future directions

Galerkin variational integrators

- symplectic-momentum preserving discretization
- based on polynomial interpolation and quadrature rules
- commutation of discretization and variation: spRK and sG
- equivalence of spRK and sG for Lobatto quadrature and separable L

Numerical optimal control

- commutation of discretization and dualization for spRK and sG

Ongoing and future work

- commutation property for general symplectic integrators
- extension to (non-)holonomic systems
- adaptive variational integrator based on varying order