Optimal feedback control, first-order PDE systems, and obstacle problems

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Basic problem

$$\text{Minimize in } \mathbf{u}(s) \in K: \quad I(\mathbf{u}) = \int_0^T F(\mathbf{x}(s), \mathbf{u}(s))\, ds (+g(\mathbf{x}(T)))$$

subject to

$$\mathbf{x}'(s) = \mathbf{f}(\mathbf{x}(s), \mathbf{u}(s)) \text{ in } (0, T), \quad \mathbf{x}(0) = \mathbf{x}_0, \mathbf{x}(s) \in \Omega$$

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- $\mathbf{f} : \Omega \times K \to \mathbb{R}^N$, map providing the state system.

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Optimal feedback map:

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Suppose $\mathbf{u}(s; t, \mathbf{y})$ for $s \in (t, T)$ is the optimal solution.
Then we take

$$\mathbf{U}(t, \mathbf{y}) \equiv \mathbf{u}(t; t, \mathbf{y})$$

Fundamental property, and objective

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Proposition

Suppose the map $\mathbf{U}(t, \mathbf{y})$ *is known, and let* $(\mathbf{x}(t), \mathbf{u}(t))$ *be an optimal pair for the control problem. Then* $\mathbf{u}(t) = \mathbf{U}(t, \mathbf{x}(t))$.

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IN PLAIN WORDS

Knowing $\mathbf{U}(t, \mathbf{y})$ will let you adjust to disturbances during the control process.

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IMPORTANT GOAL

How can one compute (approximate), in advance, the optimal feedback map $\mathbf{U}(t, \mathbf{y})$, so that measurements on the state $\mathbf{x}(t)$ would lead to the optimal control $\mathbf{u}(t) = \mathbf{U}(t, \mathbf{x}(t))$ without solving any optimal control problem?

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Important fact

Value function v must be a (viscosity) solution of

$$v_t(t, \mathbf{x}) + H(\nabla v(t, \mathbf{x}), \mathbf{x}) = 0 \text{ in } (0, T) \times \mathbb{R}^N, \quad v(T, \mathbf{x}) = 0.$$

Once v is known,

$$\mathbf{U}(t, \mathbf{x}) = \text{argmin}_{\mathbf{u} \in K} \{F(\mathbf{x}, \mathbf{u}) + \nabla v(t, \mathbf{x}) \cdot \mathbf{f}(\mathbf{x}, \mathbf{u})\}.$$

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Main difficulties: computation of H, and approximation of a fully non-linear PDE.

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$$\textit{Integrand}: \quad \phi(\mathbf{x}, \xi) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R},$$

$$\text{Minimize in } \mathbf{x}(s): \quad \int_t^T \phi(\mathbf{x}(s), \mathbf{x}'(s))\, ds, \quad \mathbf{x}(t) = \mathbf{y},$$

$$\mathbf{x}(s) \equiv \mathbf{X}(s; t, \mathbf{y}), \text{ optimal.}$$

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Under regularity and convexity assumptions

$$-\left[\phi_\xi(\mathbf{x}(s),\mathbf{x}'(s))\right]'+\phi_{\mathbf{x}}(\mathbf{x}(s),\mathbf{x}'(s))=0 \text{ in } (t,T),$$

$$\mathbf{x}(t)=\mathbf{y},\phi_\xi(\mathbf{x}(T),\mathbf{x}'(T))=0.$$

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$$\textit{Basic idea}: \quad \mathbf{x}'(s) = \mathbf{v}(s, \mathbf{x}(s)) \text{ in } (t, T), \quad \mathbf{x}(t) = \mathbf{y},$$

$$\nabla\mathbf{v}\,\mathbf{v} + \mathbf{v}_s + \phi_{\xi\xi}(\mathbf{x}, \mathbf{v})^{-1}\left(\phi_{\xi\mathbf{x}}(\mathbf{x}, \mathbf{v})\mathbf{v} - \phi_{\mathbf{x}}(\mathbf{x}, \mathbf{v})\right) = 0,$$

$$\phi_\xi(\mathbf{x}, \mathbf{v}(T, \mathbf{x})) = 0.$$

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Play, formally, with optimality conditions

$$\mathbf{u}(t, \mathbf{x}) \longrightarrow \mathbf{x}(t),$$
$$\mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t, \mathbf{x}(t)) \text{ in } (\tau, T), \quad \mathbf{x}(\tau) = \mathbf{z},$$
$$\mathbf{U}(t, \mathbf{x}) \approx \mathbf{u}(t, \mathbf{x}) \longrightarrow \mathbf{X}(t) \approx \mathbf{x}(t), \mathbf{X}(0) = 0,$$
$$\mathbf{x}' + \epsilon \mathbf{X}' = \mathbf{f}(\mathbf{x} + \epsilon \mathbf{X}, \mathbf{u}(t, \mathbf{x} + \epsilon \mathbf{X}) + \epsilon \mathbf{U}(t, \mathbf{x} + \epsilon \mathbf{X})).$$

Optimality conditions (cont'd)

$$\mathbf{X}' = (\mathbf{f_x} + \mathbf{f_u}\nabla\mathbf{u})\,\mathbf{X} + \mathbf{f_u}\mathbf{U} \text{ in } (\tau, T), \quad \mathbf{X}(\tau) = 0,$$

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Cost functional (in a similar way)

$$\int_\tau^T \left[(F_\mathbf{x} + F_\mathbf{u}\nabla\mathbf{u})\,\mathbf{X} + F_\mathbf{u}\mathbf{U}\right] dt.$$

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Costate $\mathbf{p}(t)$: $\mathbf{p}(T) = 0$

$$\mathbf{p}' + \mathbf{p}[\mathbf{f_x}(\mathbf{x}, \mathbf{u}(t, \mathbf{x})) + \mathbf{f_u}(\mathbf{x}, \mathbf{u}(t, \mathbf{x}))\nabla\mathbf{u}(t, \mathbf{x})]$$
$$= F_\mathbf{x}(\mathbf{x}, \mathbf{u}(t, \mathbf{x})) + F_\mathbf{u}(\mathbf{x}, \mathbf{u}(t, \mathbf{x}))\nabla\mathbf{u}(t, \mathbf{x}) \text{ in } (\tau, T),$$

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Introduce costate, and integrate by parts

$$\int_\tau^T [(F_\mathbf{x} + F_\mathbf{u}\nabla\mathbf{u})\,\mathbf{X} + F_\mathbf{u}\mathbf{U}]\;dt = \int_\tau^T (F_\mathbf{u} - \mathbf{p}\mathbf{f_u})\mathbf{U}\,dt.$$

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Conclusion:

1. $F_\mathbf{u} - \mathbf{p}\mathbf{f_u} \equiv 0$: $\mathbf{u}$, equilibrium;

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2. if integral < 0 for a particular $\mathbf{U}$, then descent direction.

Conclusion and reinterpretation (more formal)

Write costate $\mathbf{p}(t)$ in feedback form $\mathbf{p}(t) = \mathbf{p}(t, \mathbf{x}(t))$, so that

$$\mathbf{p}'(t) = \mathbf{p}_t(t, \mathbf{x}) + \nabla \mathbf{p}(t, \mathbf{x})\mathbf{x}' = \mathbf{p}_t(t, \mathbf{x}) + \nabla \mathbf{p}(t, \mathbf{x})\mathbf{f}(\mathbf{x}, \mathbf{u}(t, \mathbf{x})).$$

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Equation for costate $\mathbf{p}(t,\mathbf{x})$ in $(0,T)\times\mathbb{R}^N$: $\mathbf{p}(T,\mathbf{x})=0$

$$\mathbf{p}_t(t,\mathbf{x})+\nabla\mathbf{p}(t,\mathbf{x})\mathbf{f}(\mathbf{x},\mathbf{u}(t,\mathbf{x}))$$
$$+\mathbf{p}(t,\mathbf{x})[\mathbf{f}_{\mathbf{x}}(\mathbf{x},\mathbf{u}(t,\mathbf{x}))+\mathbf{f}_{\mathbf{u}}(\mathbf{x},\mathbf{u}(t,\mathbf{x}))\nabla\mathbf{u}(t,\mathbf{x})]$$
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RESULT

Start with $\mathbf{u}(t,\mathbf{x})$, compute costate $\mathbf{p}(t,\mathbf{x})$:

$$\begin{aligned}&\mathbf{p}_t(t,\mathbf{x})+\nabla\mathbf{p}(t,\mathbf{x})\mathbf{f}(\mathbf{x},\mathbf{u}(t,\mathbf{x}))+\mathbf{p}(t,\mathbf{x})\nabla_\mathbf{x}[\mathbf{f}(\mathbf{x},\mathbf{u}(t,\mathbf{x}))]\\ &=\nabla_\mathbf{x}[F(\mathbf{x},\mathbf{u}(t,\mathbf{x}))] \text{ in } (0,T)\times\mathbb{R}^N,\quad \mathbf{p}(T,\mathbf{x})\equiv 0.\end{aligned}$$

If $F_\mathbf{u}(\mathbf{x},\mathbf{u}(t,\mathbf{x}))-\mathbf{p}(t,\mathbf{x})\mathbf{f}_\mathbf{u}(\mathbf{x},\mathbf{u}(t,\mathbf{x}))=0$, then $\mathbf{u}(t,\mathbf{x})=\mathbf{U}(t,\mathbf{x})$ is an equilibrium mapping for the feedback problem .

Descent direction

$$\nabla I(\mathbf{u})(t,\mathbf{x}) \equiv F_{\mathbf{u}}(\mathbf{x},\mathbf{u}(t,\mathbf{x})) - \mathbf{p}(t,\mathbf{x})\mathbf{f}_{\mathbf{u}}(\mathbf{x},\mathbf{u}(t,\mathbf{x})),$$

$$\text{Derivative of cost at } \mathbf{u}: \quad \int_0^T \nabla I(\mathbf{u})(t,\mathbf{x}(t))\mathbf{U}(t,\mathbf{x}(t))\,dt < 0,$$

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Descent direction at **u**

Start with **u**; compute costate **p**; solve the problem

$$-\Delta\mathbf{U}(t,\mathbf{x}) + \nabla I(\mathbf{u})(t,\mathbf{x}) = 0 \text{ in } \mathbb{R}^N$$

for every $t \in [0, T]$. $\mathbf{U}(t,\mathbf{x})$ is a descent direction for **u** in an average sense: for every t,

$$\int_{\mathbb{R}^N} \nabla I(\mathbf{u})(t,\mathbf{x})\mathbf{U}(t,\mathbf{x})\,d\mathbf{x} (= -\int_{\mathbb{R}^N} |\nabla\mathbf{U}|^2\,d\mathbf{x}) < 0.$$

Numerical scheme

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1. Initialization. Take any initial $\mathbf{u}_0(t, \mathbf{x}) \in K$.

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      to determine the optimal solution $\mathbf{U}_j(t, \mathbf{x})$.

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    4. Update $\mathbf{u}_j$ to $\mathbf{u}_j + \epsilon\mathbf{U}_j$ for some small ϵ.

An LQR example (R. Font)

$$I(u) = \frac{1}{2}\int_0^T (x(t)^2 + u(t)^2)\,dt, \quad x'(t) = -x(t) + u(t).$$

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$$u(t,x) = \frac{1-\sqrt{2}+\exp(a)+\sqrt{2}\exp(a)}{1+\exp(a)}x,$$

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Iterative procedure

1. the linear, first-order PDE:

$$p_t + (u-x)p_x + (u_x - 1)p = x + uu_x \text{ in } (0,T)\times\mathbb{R}, p(T,x) = 0;$$

2. the elliptic problem, for each time slice,

$$-U_{xx} + u - p = 0 \text{ in } \mathbb{R}.$$

Numerical results

ε	iter	$\lvert I - I_\varepsilon \rvert$	$\lvert\lvert u - u_\varepsilon \rvert\rvert_\infty$	$\lvert\lvert x - x_\varepsilon \rvert\rvert_\infty$
0.001	946	$2.1876 \cdot 10^{-8}$	$9.7579 \cdot 10^{-4}$	$2.0496 \cdot 10^{-4}$
0.01	289	$1.8454 \cdot 10^{-7}$	0.0015	$1.9809 \cdot 10^{-4}$
0.1	51	$1.0072 \cdot 10^{-5}$	0.0052	0.0014
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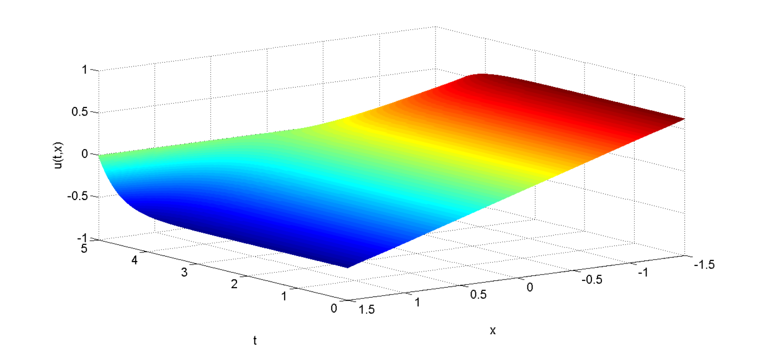

Restrictions on the control

$$I(u) = \int_0^T (x(t)^2 + u(t)^2)\, dt, \quad x' = u, \quad |u(t)| \leq 0.6.$$

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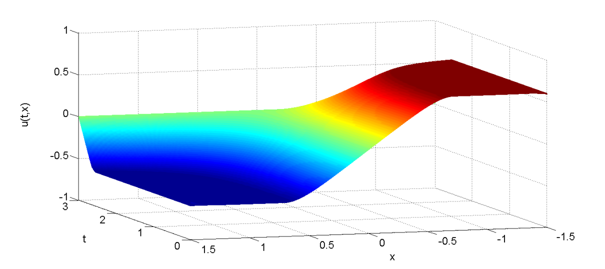
$$p_t + up_x + u_x p = 2(x + uu_x) \text{ in } (0, T) \times \mathbb{R}, \quad p(T, x) = 0;$$

2. the obstacle problem: minimize (for each fixed t) in $U(t, x) \in K$

$$\int_{\mathbb{R}} \left(\frac{1}{2} |\nabla U - \nabla u|^2 + \nabla I(u)(U - u) \right) dx$$

with $K = [-0.6,\ 0.6]$ and $\nabla I(u) = 2u - p$.

Numerical results



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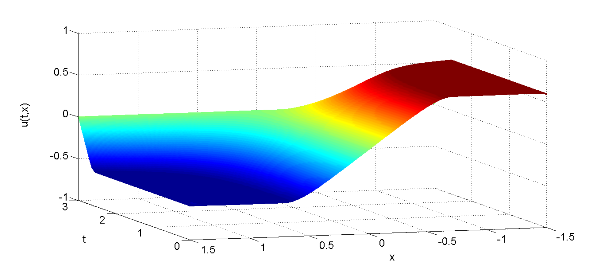

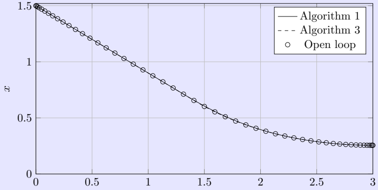

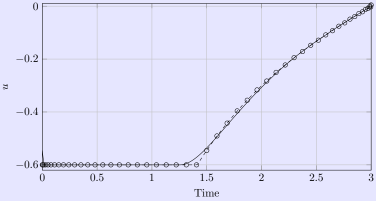

Two-dimensional situation

$$I(u) = \int_0^\infty \left(\frac{1}{2} \left(x_1(t)^2 + x_2(t)^2 \right) + u_1(t)^2 + u_2(t)^2 \right) dt$$

$$\begin{cases} \dot{x}_1(t) & = & x_1(t) - x_1(t)^3 + x_2(t) + u_1(t) \\ \dot{x}_2(t) & = & x_1(t) + x_1(t)^2 x_2(t) - x_2(t) + u_2(t) \end{cases}$$

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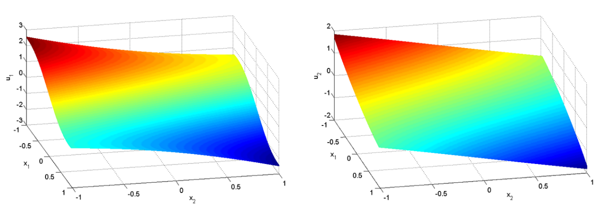
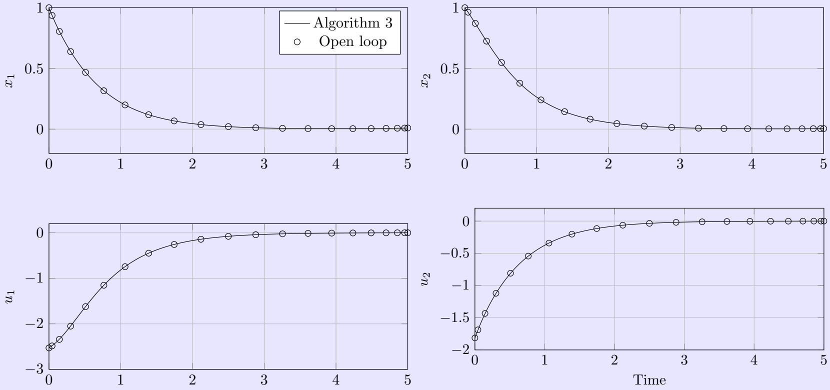


Comparison



Final example

$$I(u) = \frac{1}{2}\int_0^\infty \left(x_1(t)^2 + x_2(t)^2 + u(t)^2\right) dt$$

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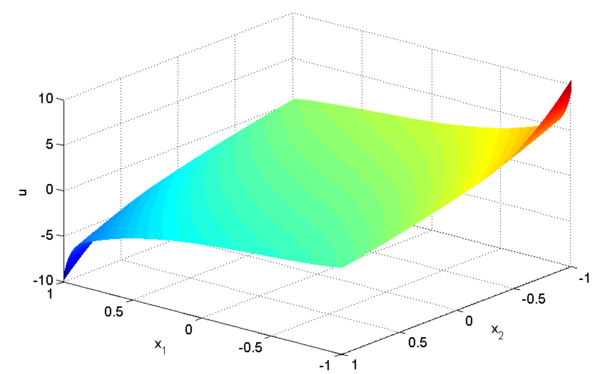

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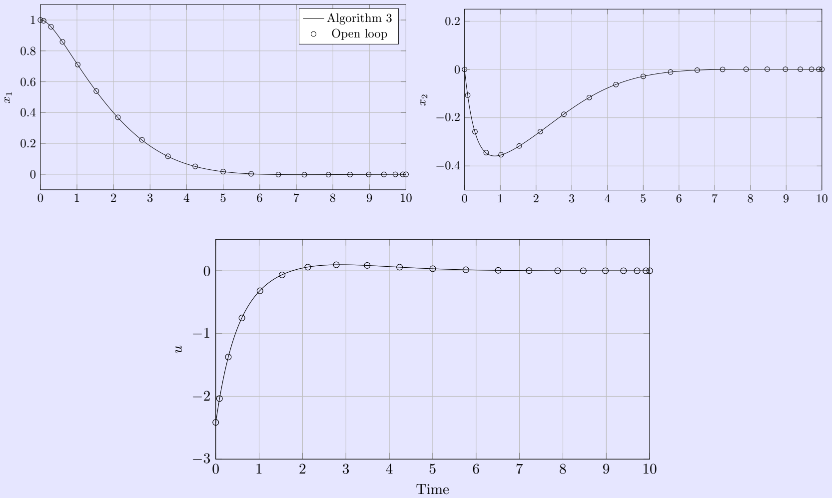
Algorithm 3
Open loop
x_1
x_2
u
Time
0
1
2
3
4
5
6
7
8
9
10
1
0.8
0.6
0.4
0.2
0.2
−0.2
−0.4
−1
−2
−3