Limit Solutions for Systems with Unbounded Controls

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July 8-12, 2014, Madrid,
AIMS Conference 2014

1. HEURISTICS

2. "LIMIT" SOLUTIONS
   - Existing notions of solutions
   - Proposed definition of Limit Solution

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(B) Investigate possible occurrence Lavrentiev phenomenon in relation to extension (A)

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- Spiking models of synaptic behaviour
- Mechanical systems using some coordinates as controls
- In general, coupled fast-slow dynamics

Underlying thought:

We can "accept" a notion of $\mathcal{L}^1$ (or *impulsive*) trajectory

PROVIDED

it is, *in some sense to be made precise*, **the LIMIT of faster and faster trajectories**

Outline

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$$x(t) = u(t) + x(0) \qquad \forall t \in [0, T] \tag{1}$$

to be a solution, which is obviously the case as soon as $x, u \in AC$ (= absolutely continuous).

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- **How to transform** (1) **into a definition when** $u, x \in \mathcal{L}^1$ **?**

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- **the non commutative case**

$$[g_\alpha, g_\beta] \neq 0$$

with the controls $u(\cdot)$ having bounded variation

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1. Due to $[g_\alpha, g_\beta] = 0$, by *multiple flow-box theorem* there exists a (global) coordinates'change

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References include

A. Bressan and F. Rampazzo. Impulsive control systems with commutative vector fields. J. Optim. Theory Appl., 71, p.67-83, (1991).

A.V. Sarychev. Nonlinear systems with impulsive and generalized function controls,vol. 9 of Progr. Systems Control Theory, p. 244-257, (1991).

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$$t'(s) = \varphi_0'(s)$$
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Namely: one bridges the jumps of u and parameterize them on s-subintervals where time $t(s)(= \varphi_0(s))$ is constant.

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$$t \to x(t) := y \circ \varphi_0^{-1}(t)$$

is called the **graph-completion solution** corresponting to the graph completion (φ_0, φ) of u. It is set-valued on a countable subset of $[a, b]$.

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single-valued version: If $\sigma : [0, T] \to [0, 1]$ is a *Clock*, i.e. $\sigma(t) \in (\varphi_0, \varphi)^{\leftarrow}(t, u(t))$, we say that

$$t \to x := y \circ \sigma(t)$$

is a **single-valued graph-completion solution.**

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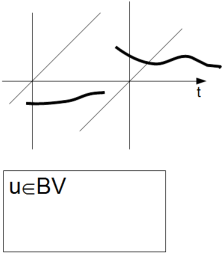

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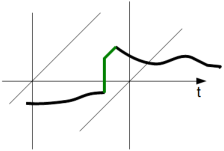

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a
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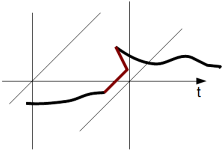

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Another
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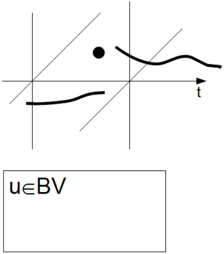

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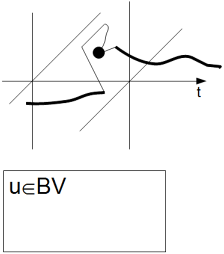

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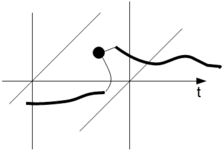

u∈BV

An incomplete list of authors who have investigated this subject:

Bressan
Bressan- Rampazzo
Bressan-Mazzola
Briani-Zidani
Pereira-Vinter
Miller
Motta-Rampazzo
Camilli-Falcone
Motta-Sartori
Sarychev
Silva
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Zabic-Wolenski

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- x **single-valued at each** t;
- **existence** of an output (and possibly *uniqueness*) for a given input u (and v),
- former definitions of solution for impulsive systems **subsumed** by this extended notion

"LIMIT SOLUTIONS"

M.S. Aronna and F. Rampazzo.
$\mathcal{L}^1$ **limit solutions for control systems**

(accepted on JDE)

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Definition

- A $\mathcal{L}^1$ map $x : [a,b] \to \mathcal{R}^n$ is a **LIMIT SOLUTION** if, *for every* $\tau \in [a,b]$, there exists a sequence of absolutely continuous controls (u_k^τ) such that
$|(x_k^\tau, u_k^\tau)(\tau) - (x,u)(\tau)| + \|(x_k^\tau, u_k^\tau) - (x,u)\|_1 \to 0$,

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- **SIMPLE LIMIT SOLUTION**: if (u_k^τ) can be chosen independently of τ, i.e. $(u_k^\tau) = (u_k)$.
- **BV-SIMPLE LIMIT SOLUTION** if the approximating inputs u_k have **equibounded variation**.

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Theorem

- **Existence and uniqueness** For every control $u \in \mathcal{L}^1$ (and every $v \in L^1$) there **exists a unique limit solution** of

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- **Continuous dependence**: for every $\tau \in [a, b]$ one has
$$|x_1(\tau) - x_2(\tau)| + \|x_1 - x_2\|_1 \leq$$
$$M\Big[|\bar{x}_1 - \bar{x}_2| + |u_1(a) - u_2(a)| + |u_1(t) - u_2(t)| + \|u_1 - u_2\|_1\Big].$$

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A worked out example of limit solution

$$\dot{x} = xv + x\dot{u}, \quad x(0) = \bar{x},$$

on the interval $[0,1]$, with $v(t) := \chi_{[0,1/2[}$
Consider the $\mathcal{L}^1$ control

$$u(t) := \begin{cases} (-1)^{k+1}, & \text{for } t \in [1-\frac{1}{k}, 1-\frac{1}{k+1}[, \ k \in \mathcal{N}, \\ 0, & \text{for } t = 1. \end{cases}$$

The *limit solution* x is given by

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A worked out example of limit solution

$$\dot{x} = xv + x\dot{u}, \quad x(0) = \bar{x},$$

on the interval $[0,1]$, with $v(t) := \chi_{[0,1/2[}$
Consider the $\mathcal{L}^1$ control

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Notice that both u and x have infinitely many discontinuities, unbounded variation, and are defined everywhere.

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(This is not straightforward: consider e.g. a BV map with a dense set of discontinuities)

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Let U have the Whitney property.
For any control pair

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Many thanks for your patience

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$u_k(t) := (k^{-1/2}\cos kt - 1, k^{-1/2}\sin kt)$ generates the trajectory

$$x_k(t) = (k^{-1/2}\cos kt - 1, k^{-1/2}\sin kt, -t + k^{-1}\sin kt)^t.$$

$$x_k(t) to \hat{x}(t) := (0, 0, -t)^t$$

so $\hat{x}$ is a (simple) limit solution. **In particular** $\hat{x} \neq x_{\mathcal{C}}$

Notice that $Var(u_k) \doteq \int_0^1 |\dot{u}_k| dt \to +\infty$

The BUT... stuff

$$\dot{x} = g_1(x)\dot{u}_1 + g_2(x)\dot{u}_2, \quad x(0) = 0.$$

$$g_1(x) := (1, 0, x_2), \quad g_2(x) := (0, 1, -x_1), \quad \text{so } [g_1, g_2] = (0, 0, -2).$$

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so $\hat{x}$ is a (simple) limit solution. **In particular** $\hat{x} \neq x_{\mathcal{C}}$

Notice that $Var(u_k) \doteq \int_0^1 |\dot{u}_k| dt \to +\infty$ **BUT...** the *iterated integral*

$$\int_0^1 |\dot{u}_k^2 u_k^1 - \dot{u}_k^1 u_k^2| dt$$

IS BOUNDED as k **goes to** ∞.

TRUE END OF THE TALK

THANKS AGAIN