Properties of Minimizers that are not also Relaxed Minimizers

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Original Problem

Consider the Optimal Control Problem:

$$(P) \quad \begin{cases} \text{minimize } g(x(0), x(1)) \\ \text{over the } x \in W^{1,1}([0,1]; \mathbb{R}^n) \text{ s.t.} \\ \dot{x}(t) \in F(t, x(t)) \text{ a.e.} \\ (x(0), x(1)) \in C \end{cases}$$

- $(H1)$: $C \subset \mathbb{R}^n \times \mathbb{R}^n$ is closed and $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous;
- $(H2)$: $F(\cdot, \cdot)$ takes values closed sets and $F(\cdot, x)$ is measurable for each x;

- For a nominal trajectory $\bar{x}(.)$

  $(H3)$: There exist $k(.), c(.) \in L^1$, and $\varepsilon > 0$ s.t.:

$$I) \quad F(t,x) \subset F(t,x') + k(t)|x - x'|B$$

$$II) \quad F(t,x) \subset c(t)B$$

  for all $x, x' \in \bar{x}(t) + \varepsilon B$, a.e. $t \in [0,1]$.

Define the maximized Hamiltonian:

$$H(t,p,x) = \max_{v \in F(t,x)} (p \cdot v)$$

Local minimizers

An F-trajectory $\bar{x}(.)$ is a **strong** local minimizer for (P) if, for $\varepsilon > 0$,

$$g(\bar{x}(0), \bar{x}(1)) \leq g(x(0), x(1))$$

for every

$$||x(.) - \bar{x}(.)||_{L^\infty} \leq \varepsilon\,.$$

$\bar{x}(.)$ is a **weak** local minimizer if

$$g(\bar{x}(0), \bar{x}(1)) \leq g(x(0), x(1))$$

for every

$$||x(.) - \bar{x}(.)||_{W^{1,1}} \leq \varepsilon\,.$$

Existence of a Minimizer

Theorem: Suppose hypotheses $(H1)$, $(H2)$ and $(H3)$ and assume, furthermore, the following conditions:

- if we write $C = C_0 \times C_1$, then or C_0 either C_1 is bounded;

- the multifunction $x \mapsto F(t, x)$ has convex value a.e. $t \in [0, 1]$.

Then (P) has a minimizer.

Relaxed Control Problem

Let's introduce the relaxed problem:

$$(R) \qquad \begin{cases} \text{minimize } g(x(0), x(1)) \\ \text{over the } x \in W^{1,1}([0,1];\mathbb{R}^n) \text{ s.t.} \\ \dot{x}(t) \in \text{co}F(t, x(t)) \text{ a.e.} \\ (x(0), x(1)) \in C \end{cases}$$

Relaxation Theorem: Assume the hypotheses $(H1)-(H3)$ and take any co$F$-trajectory x and a $\delta > 0$. Then there exists an F-trajectory y such that satisfies $y(0) = x(0)$ and

$$\max_{t \in [0,1]} |y(t) - x(t)| < \delta.$$

Reachable Set Interpretation

Define the original Reachable set

$$\mathcal{R} := \{(x(0), x(1)) : \dot{x}(t) \in F(t, x(t)) \text{ a.e. } t \in [0,1]\},$$

and the relaxed Reachable set

$$\mathcal{S} := \{(x(0), x(1)) : \dot{x}(t) \in \text{co}F(t, x(t)) \text{ a.e. } t \in [0,1]\}.$$

Then it turns out that

$$\bar{\mathcal{R}} = \mathcal{S}.$$

Why is Relaxation useful?

A standard relaxation procedure concerns:

- **Step 1:** Solve the relaxed problem (which always has a solution).

- **Step 2:** Approximate the solution to the relaxed problem as closely as required to obtain a sub-optimal solution to the original problem.

Infimum Gap

The relaxation procedure works if we assume that:

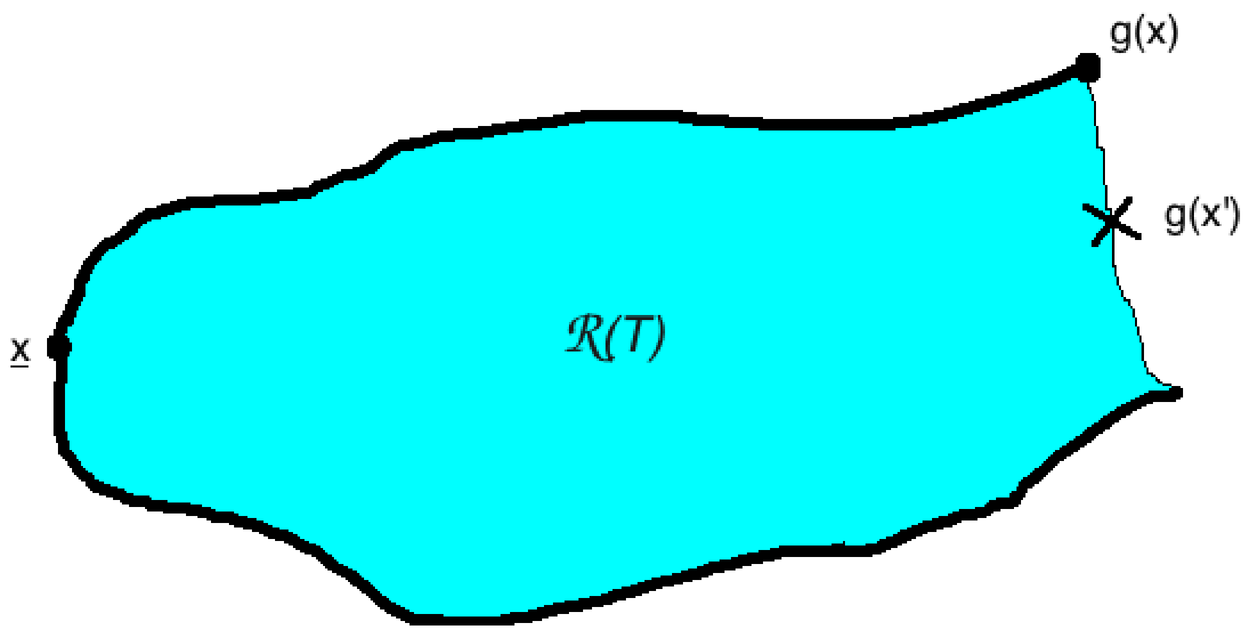
$$\inf\{P\} = \inf\{R\}.$$

But, in certain pathological situations, it could happen that

$$\inf\{P\} > \inf\{R\}. \qquad (1)$$

These situations are problematic because we cannot obtain a suboptimal minimizers for (P). Furthermore, we encounter also a breakdown of the **Dynamic Programming method and HJB approach**.

REACHABLE SET



LEGEND:

- x initial point for the dynamic
- g(x) value cost for (P)
- g(x') value cost for (P_{rel})

Infimum Gap: An Example

Consider the problem:

$$(E)\begin{cases} \text{Minimize } -x_1(1) \\ \text{over } x(.) = (x_1(.), x_2(.), x_3(.)) \text{ satisfying} \\ \dot{x}_1(t) = 0 \\ \dot{x}_2(t) = x_1(t)u(t) \\ \dot{x}_3(t) = |x_2(t)|^2 \\ u(t) \in U(t) \equiv \{-1\} \cup \{+1\} \\ x_2(0) = x_3(0) = x_3(1) = 0 \,. \end{cases}$$

It turns out that $\bar{x}(.) = (0, 0, 0)$ is an original strong local minimizer but is **not** also a relaxed minimizer. Indeed, every trajectory $\tilde{x}(.) = (\alpha, 0, 0)$ is a relaxed strong local minimizer for the bound

$$||x(.)||_{L^\infty} \leq \alpha$$

.

Abnormality of Multipliers

A trajectory $\bar{x}(.)$ is **abnormal** if there exists a couple $(\lambda, p(.))$ which satisfies some set of Necessary Conditions with $\lambda = 0$.
We relate the **"infimum gap"** (1) to **abnormality** of multipliers.

Assume "infimum gap" (1). Then
(Type A): Any **original** minimizers for (P) is abnormal.
(Type B): Any **relaxed** trajectory which satisfies (1) is abnormal.

Some Results

Type A Theorem (Palladino-Vinter), 2013
Let $\bar{x}(\cdot)$ be a strong local minimizing $F-$trajectory for (P). Assume $(H1)$, $(H2)$ and $(H3)$.
Suppose that $\bar{x}(.)$ is **not** also a minimizer for (R).

Then there exists $p(\cdot) \in W^{1,1}([0,1];\mathbb{R}^n)$ such that:

$i)$ $((\lambda = 0), p(\cdot)) \neq (0,0)$;

$ii)$ $(-\dot{p}(t), \dot{\bar{x}}(t)) \in \text{co}\partial_{x,p}H(t, \bar{x}(t), p(t))$ a.e.;

$iii)$ $(p(0), -p(1)) \in (\lambda = 0)\partial g(\bar{x}(0), \bar{x}(1)) + N_C(\bar{x}(0), \bar{x}(1))$.

Type B Theorem (Palladino-Vinter), 2013
Let $\bar{x}(\cdot)$ be a feasible co$F-$trajectory. Assume $(H1)$, $(H2)$ and $(H3)$. Suppose also

$(H4)$: there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$g(x(0), x(1)) > g(\bar{x}(0), \bar{x}(1)) + \delta$$

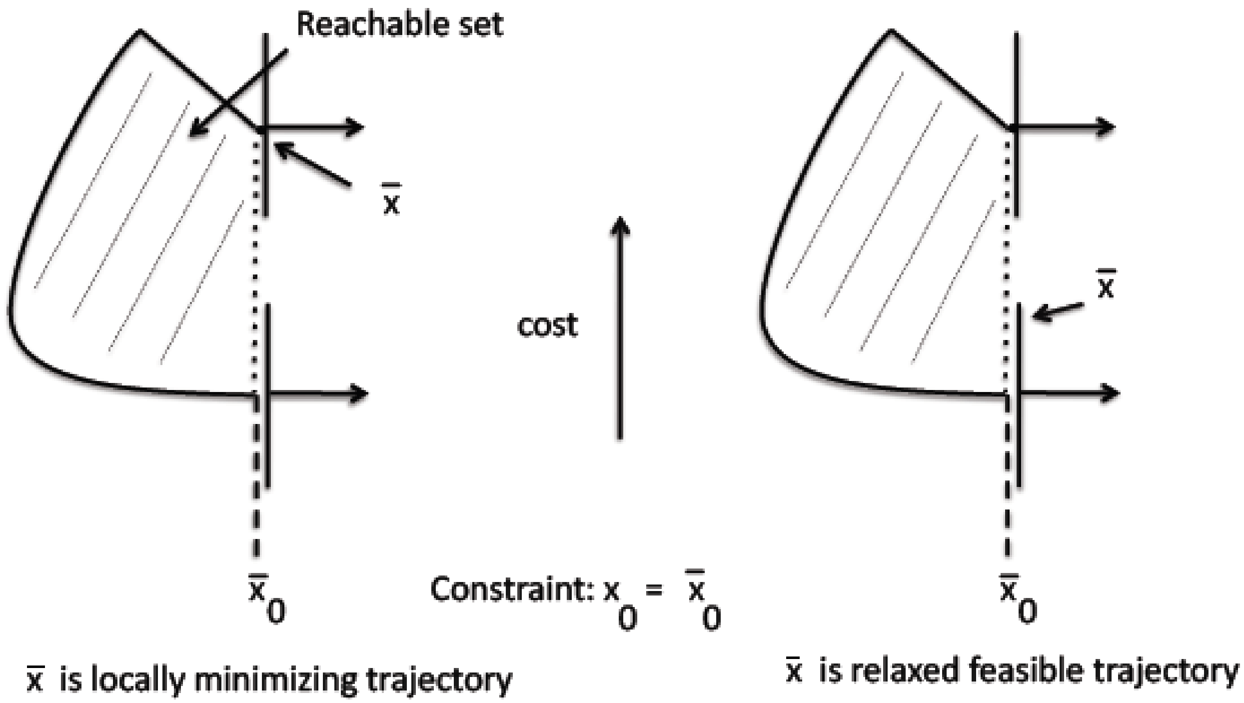
for every feasible $F-$trajectory $x(\cdot)$ such that

$$||x(\cdot) - \bar{x}(\cdot)||_\infty \leq \varepsilon.$$

Then there exists $p(\cdot) \in W^{1,1}([0,1]; \mathbb{R}^n)$ such that
$i)$ $((\lambda = 0), p(\cdot)) \neq (0, 0)$;

$ii)$ $(-\dot{p}(t), \dot{\bar{x}}(t)) \in \text{co}\partial_{x,p} H(t, \bar{x}(t), p(t))$ a.e.;

$iii)$ $(p(0), -p(1)) \in (\lambda = 0)\partial g(\bar{x}(0), \bar{x}(1)) + N_C(\bar{x}(0), \bar{x}(1))$.



Type A Theorem

Type B Theorem

Sketch of the Proof for Type B Th.

- Use the Relaxation Theorem to define an original (non feasible) sequence $x_i \to \bar{x}$ in L^∞.

- Construct a sequence of perturbed problems (respect to x_i) and use the Stegall's Variational Principle.

- Pass to the limit in the Clarke's Hamiltonian N. C.

Type B Th. $\Longrightarrow$ **Type A Th.**

Remarks

If we look at the **contrapositive** statement of the $Type\ A \quad Theorem$, it follows that normality of the minimizer implies the equality

$$\inf\{P\} = \inf\{R\}.$$

Hence, **normality** of a local minimizer is a sufficient condition in order to carry out relaxation procedure of numerical schemes for control problems.

Theory easily adapts to allow **state constraints** of the form $h(t, x(t)) \leq 0$ for every $t \in [0, 1]$.

Research Questions

- Are the theorems true if we consider **weak neighborhood** as

$$||x(.) - \bar{x}(.)||_{W^{1,1}} \leq \varepsilon \,?$$

- Are they valid also for other N. C. involving different **adjoint equations** like

$$\dot{p}(t) \in \text{co}\{q : (q, p(t)) \in N_{\text{Gr}F(t,.)}(\bar{x}(t), \dot{\bar{x}}(t))\}$$

or

$$-\dot{p}(t) \in \text{co}\{q : (q, \dot{\bar{x}}(t)) \in \partial_{x,p} H(t, \bar{x}(t), p(t))\} \,?$$

Concerning the above open questions, **Ioffe** claimed:

'Suppose that $\bar{x}(.)$ is a local $W^{1,1}$ minimizer which is normal with respect to

$$\dot{p}(t) \in \text{co}\{q : (q, p(t)) \in N_{\text{Gr}F(t,.)}(\bar{x}(t), \dot{\bar{x}}(t))\}$$

Then $\bar{x}(.)$ is also a $W^{1,1}$-relaxed minimizer'.

(A. D. Ioffe - Euler Lagrange and Hamiltonian Formalisms in Dynamic Optimization, 1997)'

This claim, as stated, is incorrect. This is illustrated by the following counter-example . . .

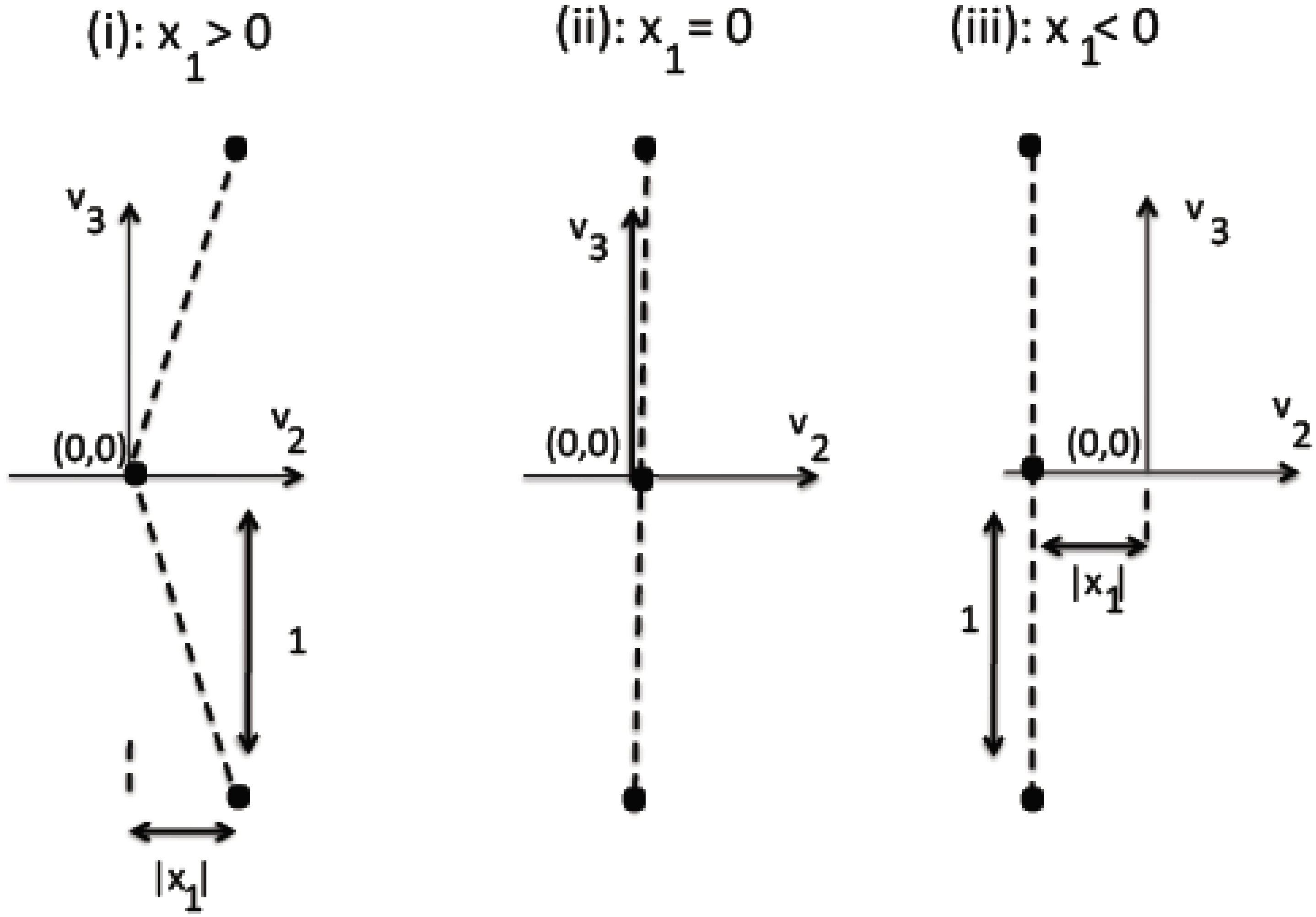
In fact, our work reveals, to correct Ioffe's assertion, we need to consider **normality w.r.t. the Hamiltonian inclusion**, and consider **local optimality w.r.t. the L^∞ norm**.

Counterexample

(R. Vinter), 2013

Consider the probem:

$$
(E)\left\{\begin{array}{l}
\text{Minimize } \left(\frac{1}{2}x_1(1) - x_2(1)\right) \\[2ex]
\text{over abs. cont. } x(.) = (x_1(.), x_2(.), x_3(.)) \text{ s.t.} \\[2ex]
\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} \in \begin{bmatrix} 0 \\ x_1(t) \\ 1 \end{bmatrix} \cup \begin{bmatrix} 0 \\ x_1(t) \\ -1 \end{bmatrix} \cup \begin{bmatrix} 0 \\ \min\{x_1(t), 0\} \\ 0 \end{bmatrix} \\[2ex]
(x_1(0), x_2(0), x_3(0)) \in [0, \infty) \times \{0\} \times \{0\}, \\[2ex]
(x_1(1), x_2(1), x_3(1)) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}
\end{array}\right.
$$



(v_2, v_3) coordinates of points (v_1, v_2, v_3) ε F(x)

It turns out that:

- Since (E) is free endpoint constraint, then the solutions are **normal**;

- $\bar{x}(.) = (0,0,0)$ is a weak local minimizer for (E) with $||x(.)||_{W^{1,1}} \leq \varepsilon < 1/2$;

- The co$F$-trajectory $\tilde{x}(.)$, determined by $\tilde{x}(0) = (\varepsilon/2\sqrt{3}, 0, 0)$ and $\dot{\tilde{x}}(t) = (0, \varepsilon/2\sqrt{3}, 0)$ is feasible and yields a cost **less** than 0.

Then $\bar{x}(.) = (0,0,0)$ is **not** also a relaxed minimizer.

To Summarize...

$$\begin{array}{c} Normality\ for\ (P)\ with \\ Fully\ Convexified\ Hamiltonian\ N.C. \end{array} \Rightarrow \begin{array}{c} \mathsf{inf}\{P\} = \mathsf{inf}\{R\} \\ in\ the\ L^{\infty}\ norm \end{array}$$

$$\begin{array}{c} Normality\ for\ (P)\ with \\ Partially\ Convexified\ Hamiltonian\ N.C. \end{array} \not\Rightarrow \begin{array}{c} \mathsf{inf}\{P\} = \mathsf{inf}\{R\} \\ in\ the\ W^{1,1}\ norm \end{array}$$

$$\begin{array}{c} Normality\ for\ (P)\ with \\ Extended\ Euler\ Lagrange\ N.C. \end{array} \not\Rightarrow \begin{array}{c} \mathsf{inf}\{P\} = \mathsf{inf}\{R\} \\ in\ the\ W^{1,1}\ norm \end{array}$$

Open Research Question

Consider the problem:

$$(P) \begin{cases} \text{minimize}\, g(x(0), x(1)) \\ \text{over the absolutely continuos arcs}\, x(\cdot) \\ \text{and the measurable control functions}\, u(\cdot) \text{ s.t.} \\ \dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. } t \in [0, 1] \\ u(t) \in U(t) \quad \text{a.e. } t \in [0, 1] \\ (x(0), x(1)) \in C \end{cases}$$

Warga proved a Type B Theorem for a Nonsmooth Maximum principle.

Is the Type A Theorem also true?

If we assume the dynamic has the **structure**

$$f(t,x,u) := A(t,x) + B(t,x)u$$

with

$$det(B(t,x)B^T(t,x)) \neq 0,$$

for every x, a.e. $t \in [0,1]$, then a **Type A theorem** is valid for the Nonsmooth P.M.P. with adjoint equation

$$-\dot{p}(t) \in \text{co}\, \partial_x h(t, \bar{x}(t), \bar{u}(t), p(t)) \quad \text{a.e.}$$

References

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- **R. B. Vinter** - "The Hamiltonian Inclusion for Non-Convex Velocity Sets", submitted;

- **J. Warga** - "Optimal Control of Differential and Functional Equations", 1972;
- **J. Warga** - "Controllability, Extremality and Abnormality in Nonsmooth Optimal Control", 1983.

Thank you!