Computation of Local ISS Lyapunov Function Via Linear Programming

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ITN SADCO
Initial Training Network
Sensitivity Analysis for Deterministic Controller Design

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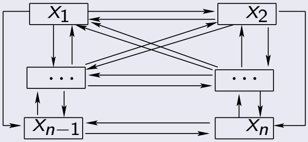
Outline

Motivation

Estimate the domain of attraction of interconnected systems



$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, \cdots, x_n), \\ \quad \vdots \\ \dot{x}_n = f_n(x_1, x_2, \cdots, x_n), \end{cases} \qquad x_i \in \mathbb{R}^{n_i}, \ \sum_{i=1}^{n} n_i = N,$$

f_i is Lipschitz continuous and $f_i(0, 0, \cdots, 0) = 0$.

Motivation

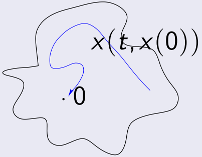
Domain of attraction

$$\begin{aligned} \dot{x} &= f(x). \\ f(0) &= 0, \quad f \text{ is Lipschitz continuous.} \end{aligned} \qquad (1)$$

If 0 is locally asymptotically stable, then the domain of attraction of 0 is defined as

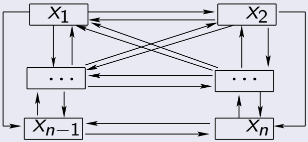
$$\mathscr{D}(0) = \left\{ x(0) \in \mathbb{R}^N : x(t, x(0)) \to 0 \quad \text{as } t \to \infty \right\}.$$

Here $x(t, x(0))$ denotes the solution of system (1).



Motivation

Estimate the domain of attraction of interconnected systems



$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, \cdots, x_n), \\ \quad \vdots \\ \dot{x}_n = f_n(x_1, x_2, \cdots, x_n), \end{cases} \qquad x_i \in \mathbb{R}^{n_i}, \ \sum_{i=1}^{n} n_i = N,$$

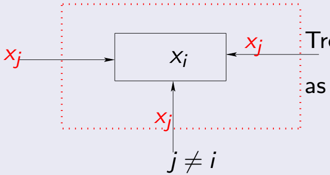
f_i is Lipschitz continuous and $f_i(0, 0, \cdots, 0) = 0$.

If subsystems are input to state stable(ISS), an effective way to estimate the domain of attraction

- using small gain theorems with subsystems' input to state stability (ISS) Lyapunov functions.

Question: How to compute local ISS Lyapunov functions for subsystems?

Subsystems



Treating effect of $x_j (j \neq i)$ as perturbation.

$$S_i: \quad \dot{x}_i = f_i(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n).$$

System with perturbation

$$\dot{x} = f(x, u), \; x \in \mathbb{R}^n, u \in \mathscr{U}_R, \tag{2}$$

Assumptions:

- $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz continuous and $f(0,0) = 0$.
- System (2) is locally input to state stable(ISS).

Notations:
$U_R := \overline{B(0,R)} \subset \mathbb{R}^m$, admissible input values $u \in U_R$ and the control input functions are $u \in \mathscr{U}_R := \{u : \mathbb{R} \to \mathbb{R}^m \text{ measurable } | \|u\|_{L_\infty} \leq R\}$, $\|u\|_{L_\infty} = ess.\sup\{\|u(t)\|_2, t \geq 0\}$.

ISS Lyapunov function

Definition (ISS Lyapunov function)

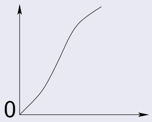
A smooth function $V : \mathscr{D} \to \mathbb{R}_{\geq 0}$, with $\mathscr{D} \subset \mathbb{R}^n$ open, is a local ISS Lyapunov function of system (2) if there exist $\rho^0 > 0$, $\rho^u > 0$, $\mathscr{K}_\infty$ functions φ_1, φ_2, α and β such that $B(0,\rho^0) \subset \mathscr{D}$ and

$$\begin{aligned} \varphi_1(\|x\|_2) &\leq V(x) \leq \varphi_2(\|x\|_2), \\ \dot{V} = <\nabla V(x), f(x,u)> &\leq -\alpha(\|x\|_2) + \beta(\|u\|_2), \end{aligned}$$

for all $\|x\|_2 \leq \rho^0$, $\|u\|_2 \leq \rho^u$.
If $\rho^0 = \rho^u = \infty$ then V is called a global ISS Lyapunov function.

$\mathscr{K}_\infty$ function

$\alpha(\cdot)$
$\beta(\cdot)$



Relation Between ISS and Robust Lyapunov Functions

Introduce two Lipschitz continuous functions $\eta_1 : \mathbb{R}^n \to \mathbb{R}^n$ and $\eta_2 : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$, then consider the auxiliary System

$$\begin{aligned} \dot{x} &= f_\eta(x,u) := f(x,u) - \eta_1(x)\eta_2(u), \quad \|u\| \leq R. \\ x(0) &= x^0, \quad f_\eta(0,u) = 0. \end{aligned}$$

Theorem

If there exists a local robust Lyapunov function $V(x)$ for the auxiliary system and

- $\exists \alpha(\cdot) \in \mathscr{K}_\infty$ such that $< \nabla V(x), f_\eta(x,u) > \leq -\alpha(\|x\|_2)$, for all $u \in \mathscr{U}_R$,
- $\exists K > 0$ and $\beta(\cdot) \in \mathscr{K}_\infty$ such that $< \nabla V(x), \eta_1(x) > \leq K (K > 0)$, and $|\eta_2(u)| \leq \beta(\|u\|_2)$,

then $V(x)$ is a local ISS Lyapunov function for the original system.

Question: How to compute such a robust Lyapunov function $V(x)$ with $\eta_1(x)$ and $\eta_2(u)$?

Relation Between ISS and Robust Lyapunov Functions

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then $V(x)$ is a local ISS Lyapunov function for the original system.

Question: How to compute such a robust Lyapunov function $V(x)$ with $\eta_1(x)$ and $\eta_2(u)$?

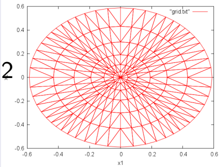
Computing Local Robust Lyapunov Functions by Linear Programming

Grids

Aim:
Compute a robust Lyapunov function $V(x)$ which is linear affine on each simplex. Such a function $V(x)$ is Lipschitz continuous.

We divide a compact set $\Omega \subset \mathbb{R}^n$ into N $n-$ simplices
$\mathscr{T} = \{\Gamma_\nu | \nu = 1, \ldots, N\}$, $\Gamma_\nu := co\{x_0, x_1, \cdots, x_n\}$, $h_x := diam(\Gamma_\nu)$, $diam(\Gamma_\nu) := \max_{x,y \in \Gamma_\nu} \|x - y\|_2$.
We divide a compact set $\Omega_u \subset U_R$ into N_u $m-$ simplices
$\mathscr{T}_u = \{\Gamma_\nu^u | \nu = 1, \ldots, N_u\}$, $\Gamma_\nu^u := co\{u_0, u_1, \cdots, u_m\}$, $h_u := diam(\Gamma_\nu^u)$,

n or $m = 2$



Clarke's subdifferential for Lipschitz continuous functions

How to extend the definition of smooth robust Lyapunov function to definition of piecewise affine robust Lyapunov function i.e. nonsmooth robust Lyapunov function?

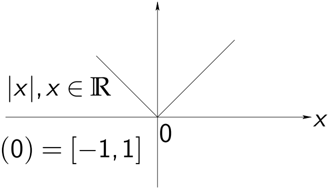
Proposition (Clarke,1998)

For a Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}$ Clarke's subdifferential satisfies

$$\partial_{Cl} V(x) = co\{\lim_{i\to\infty} \nabla V(x_i) | x_i \to x, \nabla V(x_i) \text{exists and} \lim_{i\to\infty} \nabla V(x_i) \text{ exists}\}.$$

$V(x) = |x|, x \in \mathbb{R}$

$\partial_{Cl} V(0) = [-1, 1]$



Nonsmooth robust Lyapunov function

System

$$\dot{x} = f(x,u),\ x \in \mathbb{R}^n, u \in \mathbb{R}^m,$$

$f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz continuous and $f(0,u) = 0$.

Definition (Nonsmooth robust Lyapunov function)

A Lipschitz continuous function $V : \mathscr{D} \to \mathbb{R}_{\geq 0}$, with $\mathscr{D} \subset \mathbb{R}^n$ open is said to be a local nonsmooth robust Lyapunov function of system if there exist $\rho^0 > 0$, $\rho^u > 0$, $\mathscr{K}_\infty$ functions φ_1 and φ_2, and a positive definite function α such that $B(0,\rho^0) \subset \mathscr{D}$ and

$$\begin{aligned} \varphi_1(\|x\|_2) &\leq V(x) \leq \varphi_2(\|x\|_2), \\ \sup_{u \in U_R} < \xi, f(x,u) > &\leq -\alpha(\|x\|_2), \quad \forall \xi \in \partial_{cl} V(x), \end{aligned}$$

for all $\|x\|_2 \leq \rho^0$, $\|u\|_2 \leq \rho^u$.
If $\rho^0 = \rho^u = \infty$ then V is called a global nonsmooth robust Lyapunov function.

Computing local robust Lyapunov functions by linear programming

System

$$\dot{x} = f(x, u),\ x \in \mathbb{R}^n, u \in \mathbb{R}^m,$$

Assumptions:

- $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz continuous and $f(0, u) = 0$.
- System is locally asymptotically stable at the origin uniformly in u.

Problem

How to compute a local robust Lyapunov function $V(x)$ for the system?

Linear programming based algorithm for computing piecewise affine Lyapunov functions

Previous results of Linear programming based algorithm for computing piecewise affine Lyapunov functions

1. Linear programming based algorithm for computing piecewise affine Lyapunov function was first presented in [Marinósson,2002] for ordinary differential equations.
2. Further developed in [Hafstein,2007] for systems with switching time.
3. Extended to nonlinear differential inclusions in [Baier, Grüne, Hafstein,2012].

* Gives a true Lyapunov function, i.e., not an approximation of a Lyapunov function.

Computing local robust Lyapunov functions by linear programming

Notation: $PL(\Omega)$: the space of continuous functions $V:\Omega\to\mathbb{R}$ which are linear affine on each simplex, i.e. $\nabla V_\nu := \nabla V|_{int\Gamma_\nu} \equiv const$, for all $\Gamma_\nu \in \Omega$.

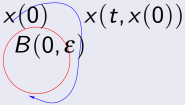
System:

$$\dot{x} = f(x,u), \quad f(0,u) = 0.$$

Assumption: System is locally asymptotically stable at the origin uniformly in u.

Aim: compute a local robust Lyapunov function $V(x) \in PL(\Omega \backslash B(0,\varepsilon))$($\varepsilon > 0$ small enough) satisfying by linear programming.

- $< \nabla V(x), f(x,u) > \leq -\|x\|_2$.



Reason for excluding $B(0,\varepsilon)$

$< \nabla V(x), f(x,u) > \leq -\|x\|_2$.

Computing robust Lyapunov functions by linear programming

Linear programming based algorithm 1

Let $\mathscr{T}^{\varepsilon} := \{\Gamma_\nu | \Gamma_\nu \cap B(0,\varepsilon) = \emptyset\} \subset \mathscr{T}$, $\varepsilon > 0$ is small enough.

1. For all vertices x_i of $\Gamma_\nu \in \mathscr{T}^{\varepsilon}$, introduce $V(x_i)$ as the variables and demand $V(x_i) \geq \|x_i\|_2 \Longrightarrow V(x) \geq \|x\|_2, x \in \Gamma_\nu \in \mathscr{T}^{\varepsilon}$.
2. For every $\Gamma_\nu \in \mathscr{T}^{\varepsilon}$, introduce the variables $C_{\nu,i}$ $(i = 1, 2, \ldots, n)$, G and require
$$|\nabla V_{\nu,i}| \leq C_{\nu,i} \leq G,$$
$\nabla V_{\nu,i}$ is the i-th component of the vector ∇V_ν.
3. For every $\Gamma_\nu \in \mathscr{T}^{\varepsilon}$, and every Γ_ν^u, demand
$$< \nabla V_\nu, f(x_i, u_j) > + \sum_{k=1}^{n} C_{\nu k} \underbrace{\left(A_x(u, h_x) + A_u(x, h_u)\right)} \leq -\|x_i\|_2,$$
compensate for interpolation errors for the points (x, u) with $x \neq x_i, u \neq u_j$, $i = 0, 1, \ldots, n$, $j = 0, 1, \ldots, m$.

Theorem (Result of linear programming based algorithm 1)

If $x \mapsto f(x,u)$ is Lipschitz continuous in x uniformly in u and $u \mapsto f(x,u)$ is similar for u, and the linear programming problem with the constraints in $1.-3.$ has a feasible solution, then the values $V(x_i)$ at all the vertices x_i of all the simplices $\Gamma_\nu \in \mathscr{T}^\varepsilon$ and the condition $V \in PL(\mathscr{T}^\varepsilon)$ uniquely define the function

$$V : \bigcup_{\Gamma_\nu \in \mathscr{T}^\varepsilon} \Gamma_\nu \to \mathbb{R},$$

satisfying

$$< \nabla V_\nu, f(x,u) > \leq -\|x\|_2$$

for $x \in \Gamma_\nu$ and $u \in \Gamma_\nu^u$. Furthermore V is a robust Lyapunov function for system.

Problem

How to compute a local ISS Lyapunov function $V(x)$ with $\eta_1(x)$ and $\eta_2(u)$ for the original system by linear programming?

Nonsmooth ISS Lyapunov function

System

$$\dot{x} = f(x,u),\ x \in \mathbb{R}^n, u \in \mathbb{R}^m,$$

$f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz continuous and $f(0,0) = 0$.

Definition (Nonsmooth ISS Lyapunov function)

A Lipschitz continuous function $V : \mathscr{D} \to \mathbb{R}_+$, with $\mathscr{D} \subset \mathbb{R}^n$ open is said to be a local nonsmooth ISS-Lyapunov function of system if there exist $\rho^0 > 0$, $\rho^u > 0$, $\mathscr{K}_\infty$ functions φ_1, φ_2, α and β such that $B(0,\rho^0) \subset \mathscr{D}$ and

$$\begin{aligned} \varphi_1(\|x\|_2) &\leq V(x) \leq \varphi_2(\|x\|_2), \\ \sup_{u \in U_R} < \xi, f(x,u) > &\leq -\alpha(\|x\|_2) + \beta(\|u\|_2), \quad \forall \xi \in \partial_{cl} V(x), \end{aligned}$$

for all $\|x\|_2 \leq \rho^0$, $\|u\|_2 \leq \rho^u$.
If $\rho^0 = \rho^u = \infty$ then V is called a global nonsmooth ISS Lyapunov function.

Computing Local ISS Lyapunov Functions by Linear Programming

Auxiliary system

Ansatz for the original system two Lipschitz continuous functions $\eta_1 : \mathbb{R}^n \to \mathbb{R}^n$ and $\eta_2 : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$,

$$\eta_1(x) = \sum_{i=0}^{n} \lambda_i \eta_1(x_i), \text{ for } x = \sum_{i=0}^{n} \lambda_i x_i \in \Gamma_\nu, \sum_{i=0}^{n} \lambda_i = 1,$$

$$\eta_2(u) = r \sum_{j=0}^{m} \mu_j \|u_j\|_2, \text{ for } u = \sum_{j=0}^{m} \mu_j u_j \in \Gamma_\nu^u \text{ and } r \geq 0, \sum_{j=0}^{m} \mu_j = 1.$$

Auxiliary system:

$$\dot{x} = f_\eta(x,u) := f(x,u) - \eta_1(x)\eta_2(u).$$

Computing Local ISS Lyapunov Functions by Linear Programming

Auxiliary system:

$$\dot{x} = f_\eta(x,u) := f(x,u) - \eta_1(x)\eta_2(u).$$

Aim: compute a local robust Lyapunov function $V(x) \in PL(\Omega \backslash B(0,\varepsilon))$ for $x \in \Gamma_\nu \in \Omega \backslash B(0,\varepsilon)$ satisfying by linear programming

- $K \geq < \nabla V(x), \eta_1(x) > \geq 1 \; (K > 0)$.
- $< \nabla V(x), f_\eta(x,u) > \leq -\|x\|_2$.

Reasons for excluding $B(0,\varepsilon)$

(i) $0 \in \partial_{cl} V(0)$ may hold which is contradictory with $< \nabla V(x), \eta_1(x) > \geq 1$.
(ii) $< \nabla V(x), f_\eta(x,u) > \leq -\|x\|_2$.

Computing Local ISS Lyapunov Functions by Linear Programming

Linear programming based algorithm 2

Linear programming problem 1:

1. $V(x)$ is positive definite. The constraints are the same as 1. in linear programming based algorithm 1.
2. $|\nabla V_{\nu,i}| \leq C_{\nu,i} \leq G$. The requirements are the same as 2. in linear programming based algorithm 1.
3. For every $\Gamma_\nu \in \mathscr{T}^\varepsilon$, and every Γ_ν^u, introduce a nonnegative variable r and demand

$$< \nabla V_\nu, f(x_i, u_j) > - r\|u_j\|_2 + \sum_{k=1}^{n} C_{\nu k}(A_x(u, h_x) + A_u(x, h_u)) \leq -\|x_i\|_2.$$

Objective function: $\min\{r + G\}$.

After getting $V \in PL(\mathscr{T}^\varepsilon)$ defined by values $V(x_i)$ at all the vertices x_i of all the simplices $\Gamma_\nu \in \mathscr{T}^\varepsilon$, consider another linear programming problem.

Computing Local ISS Lyapunov Functions by Linear Programming

Linear programming based algorithm 2

Linear programming problem 2:

4. Introduce a nonnegative variable K and variables $\eta_{1,k}(x_i)$ $(k = 1, 2, \ldots, n)$. For every $\Gamma_\nu \in \mathscr{T}^\varepsilon$, demand

$$K \geq < \nabla V_\nu, \eta_1(x_i) > \geq 1, i = 1, 2, \ldots, n,$$

$$\eta_1(x_i) = (\eta_{1,1}(x_i), \eta_{1,2}(x_i), \ldots, \eta_{1,n}(x_i)).$$

Objective function: $\min\{K\}$

Result of linear programming based algorithm 2

Consider

$$\dot{x} = f(x,u),\ x \in \mathbb{R}^n, u \in \mathscr{U}_R. \tag{3}$$

Assumptions: $f(0,0) = 0$, $x \mapsto f(x,u)$ is Lipschitz continuous in x uniformly in u, and $u \mapsto f(x,u)$ is similar.

Theorem

If the assumptions hold and the linear programming problems constructed by the algorithm 2 have feasible solutions, then the values $V(x_i)$ at all the vertices x_i of all the simplices $\Gamma_\nu \in \mathscr{T}^\varepsilon$ and the condition $V \in PL(\mathscr{T}^\varepsilon)$ uniquely define the function $V : \bigcup_{\Gamma_\nu \in \mathscr{T}^\varepsilon} \Gamma_\nu \to \mathbb{R}$. Furthermore $V(x)$ is a local ISS Lyapunov function and satisfies

$$< \nabla V_\nu, f(x,u) > \lessapprox -\|x\|_2 + rK\|u\|_2.$$

Remark: If the assumption $x \times u \mapsto f(x,u)$ is two times continuously differentiable with respect to x for every fixed u and with respect to u for every fixed x hold, then we get better computation errors.

Existence of Solutions of Linear Programming Problems

Consider

$$\dot{x} = f(x,u), \; x \in \mathbb{R}^n, u \in \mathcal{U}_R. \tag{4}$$

Assumptions: $f(0,0) = 0$, $x \mapsto f(x,u)$ is Lipschitz continuous in x uniformly in u, and $u \mapsto f(x,u)$ is similar.

Theorem

If the assumptions hold, and system (4) has a local C^2 ISS Lyapunov function $V^* : \Omega \to \mathbb{R}$ and let $\varepsilon > 0$, then there exist triangulations $\mathcal{T}^{\varepsilon}$ and $\mathcal{T}_u$ such that the linear programming problems constructed by the algorithm 2 have feasible solutions and deliver a local ISS Lyapunov function $V \in PL(\mathcal{T}^{\varepsilon})$.

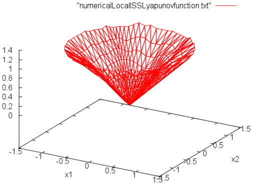
Example

Consider system adapted from [Michel, Sarabudla, Miller, 1982] is described by

$$S_1: \begin{cases} \dot{x}_1 = -x_1\left[4-(x_1^2+x_2^2)\right] + 0.1u_1, \\ \dot{x}_2 = -x_2\left[4-(x_1^2+x_2^2)\right] - 0.1u_2, \end{cases}$$

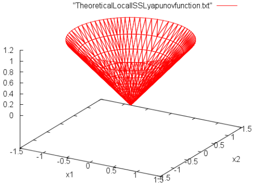
on $x = (x_1, x_2)^\top \in [-1.5, 1.5]^2$, $u = (u_1, u_2)^\top \in [-0.6, 0.6]^2$.

$\varepsilon = 0.012$
$r = 0.041$
$k = 1.000$



$V(x)$ obtained by the algorithm

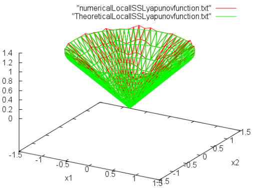
$\varepsilon = 0.012$



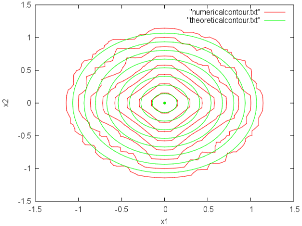
Theoretical $V_1(x)$ based on $V^* = \|x\|_2$

Difference between $V(x)$ and $V_1(x)$

$\varepsilon = 0.012$



Difference between contours of $V(x)$ and $V_1(x)$



Conclusion and Future Works

Conclusion

- A new way of computing local ISS Lyapunov functions is given.

Future works

- Consider two linear programming problems of algorithm 2 as one quadratic problem.
- Consider the problem of computing local ISS Lyapunov functions without introducing auxiliary systems.
- Estimate the domain of attraction of interconnected systems by small gain theorems with subsystems' local ISS Lyapunov functions obtained by linear programming.

Thanks for your attention!