Sensitivity-based multistep feedback MPC: from algorithm to hardware design

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SEVENTH FRAMEWORK PROGRAMME
MARIE CURIE ACTIONS

ITN SADCO
Initial Training Network
Sensitivity Analysis for Deterministic Controller Design

UNIVERSITÄT BAYREUTH

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- Introduction to MPC
- Motivations for the project
- MPC Problem Formulation + Sensitivity Analysis = SBMF
- Results & Future Work

Model Predictive Control (MPC)

- an algorithm to find $\mu : X \to U$ such that x^* is asymptotically stable for the feedback controlled system

$$x_\mu(n+1) = f(x_\mu(n), \mu(x_\mu(n)))$$

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- advantages include its capability to handle constraints on variables and the flexibility for the formulation of objective function and process model

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2. Set $\tilde{x}_0 = x(n)$ and solve the OCP:

$$\min_{x(t),u(t)} \quad m(x(t_{f_n})) + \int_{t_n}^{t_{f_n}} \ell\,(x(t), u(t))\,dt$$

$$\text{subject to} \quad \begin{aligned} &\dot{x}(t) = f(x(t), u(t)), \quad x(t_n) = \tilde{x}_0 \\ &g(x(t), u(t)) = 0 \\ &h(x(t), u(t)) \geq 0 \end{aligned}$$

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$$\min_{\substack{x0,\ldots,x_N,\\ u_0,\ldots,u_{N-1}}} \quad m(x_N) + \sum_{k=0}^{N-1} \ell_d\,(x_k, u_k)$$

subject to

$$x_{k+1} = f_d(x_k, u_k), \quad x_0 = \tilde{x}_0$$
$$g_d(x_0, \ldots, x_N, u_0, \ldots, u_{N-1}) = 0$$
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and denote optimal control sequence by $u_0^*, \ldots, u_{N-1}^*$.

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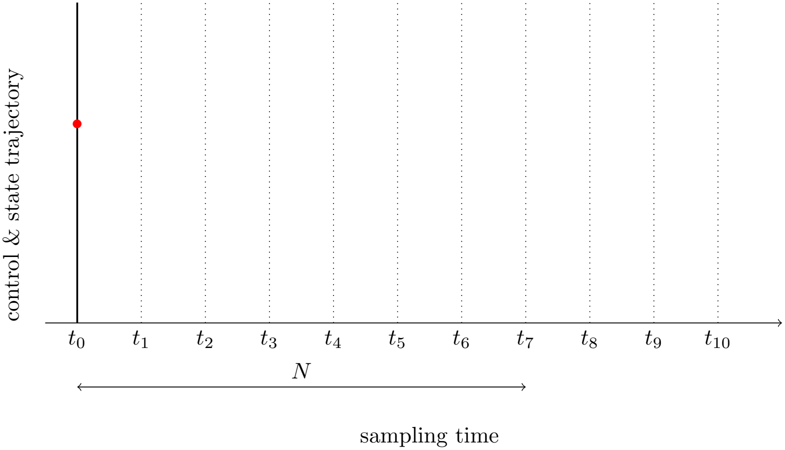
and denote optimal control sequence by $u_0^*, \ldots, u_{N-1}^*$.

3. Define MPC-feedback $\mu_N(x(n)) := u_0^*$ and use this to generate the next state.

Basic MPC Algorithm



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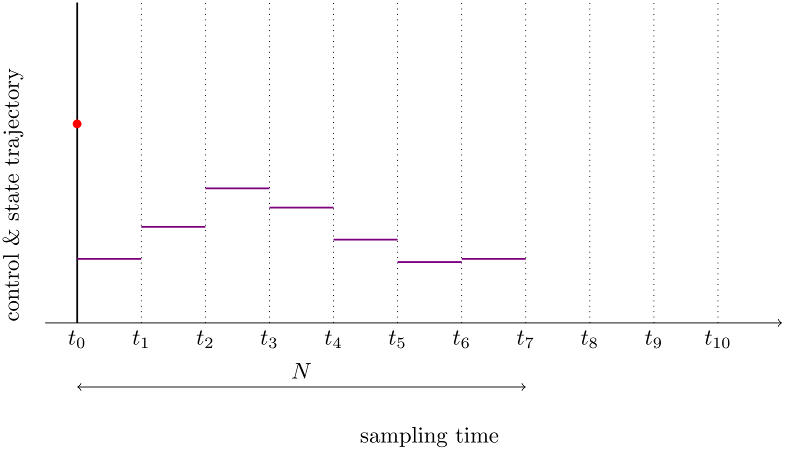

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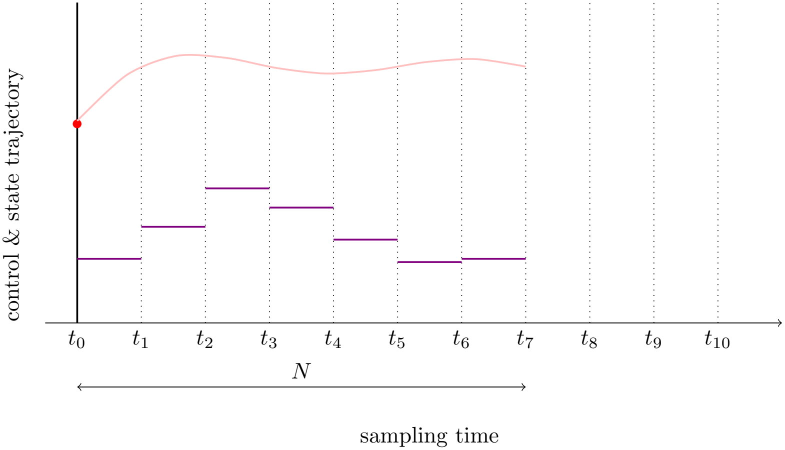

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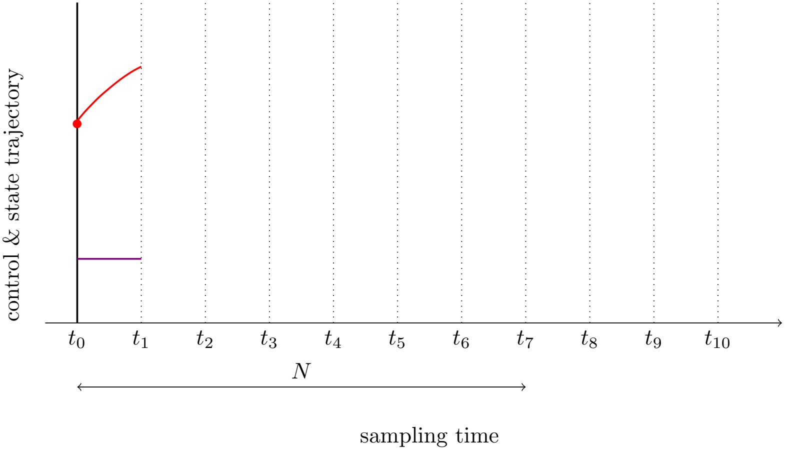

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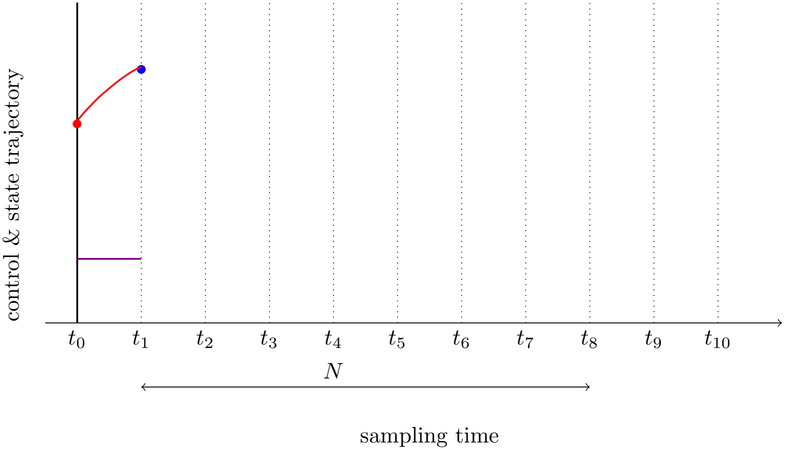

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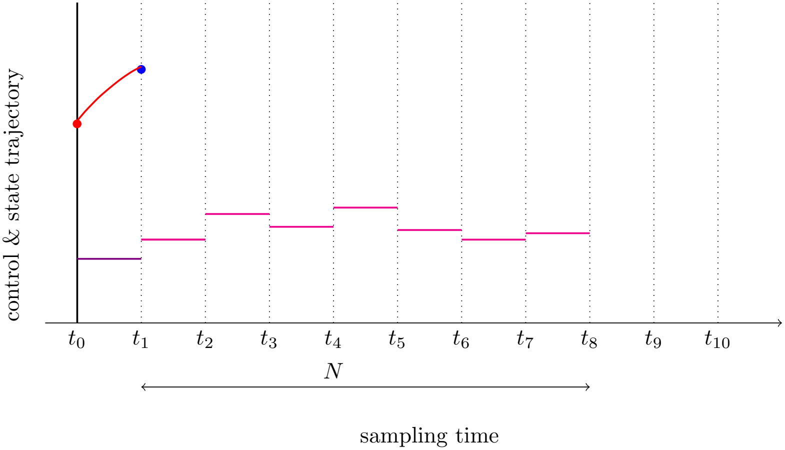

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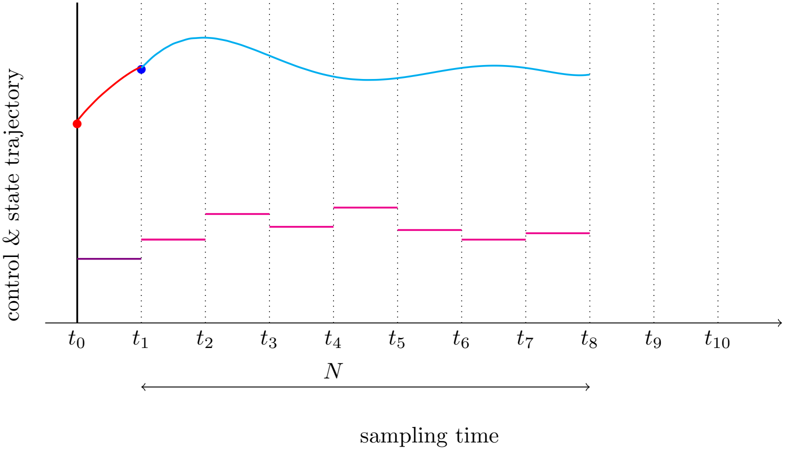

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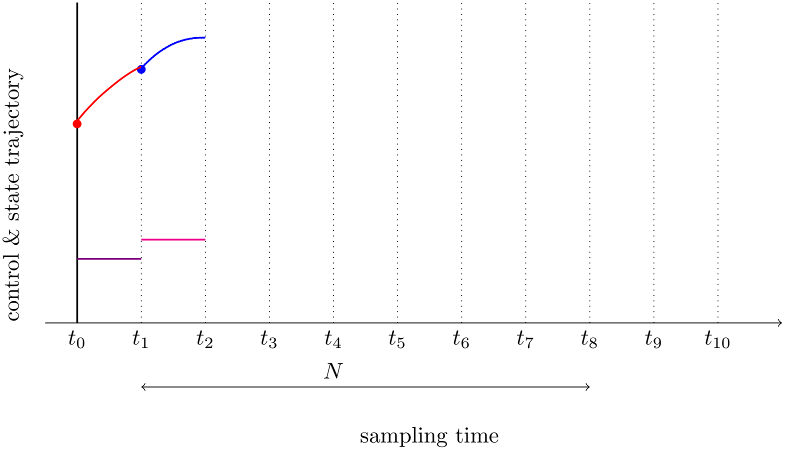

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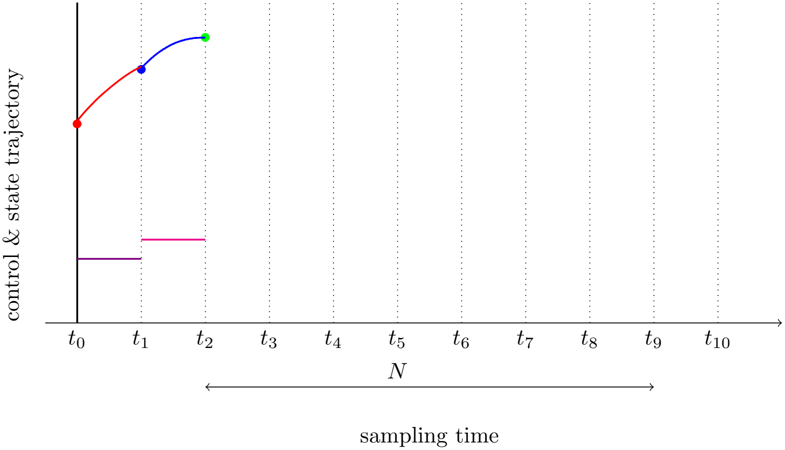

Motivation: Plant and Model 'Closeness'

- MPC - a feedback strategy

feedback if and only if there is uncertainty

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[1] Åstrom & Murray. *Feedback Systems: An introduction for scientists & engineers*. Princeton, 2008.

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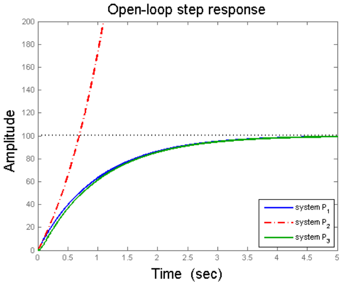
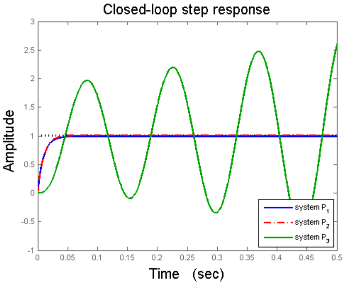
- a plant vs. mathematical model mismatch!
- 'closeness' in open-loop $\not\Rightarrow$ 'closeness' in closed-loop[1]





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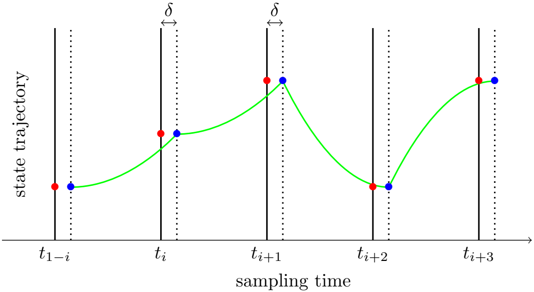
Motivation: Real-time Iteration

- solving an OCP can be very computationally intensive.
- once the current system state is measured, the OCP must be solved *on the fly!*



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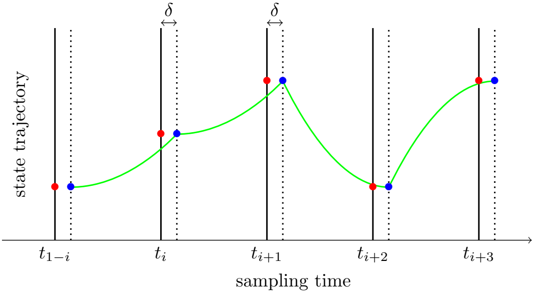
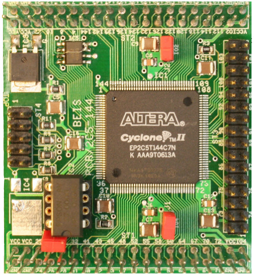
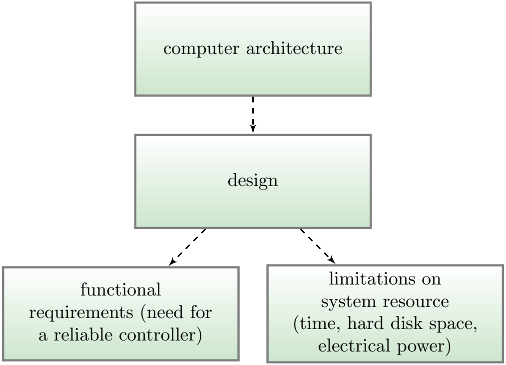


- δ is called the computational delay or latency. How can latency be reduced?

Motivation: Computer Architecture

the development of an algorithm $+$ the design of a machine





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- The power consumed by a CPU obeys the following relation

$$P \sim \alpha \ C \, V^2 \ f$$

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power consumption

capacitance

voltage

clock frequency

- How can power consumption be reduced?

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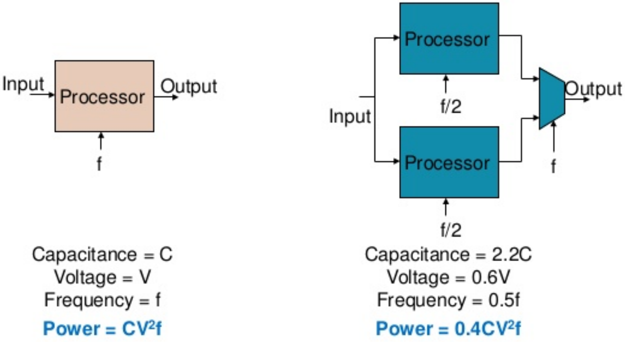
- suppose τ is the time required in to compute control action
- what happens if $\dfrac{K}{\mathsf{IPC} \cdot f} < \tau$? high performance but wastes power!
- an ideal frequency $f_{\mathsf{ideal}} = \dfrac{N}{\mathsf{IPC} \cdot \tau}$

lower clock frequency $\implies$ less power consumption

Motivation: Computer Architecture

- A key could be computer architectures that allow for parallelization and pipelining techniques.



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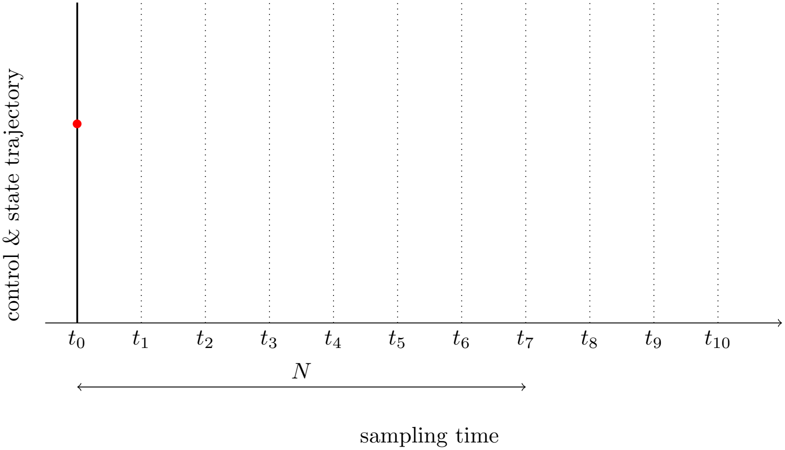
[2] Ian Phillips. *Energy Efficient Computing*. 2013

Multistep Feedback (MF) MPC

- a straightforward approach to reduce computational cost
- multistep feedback
- using more than just the first element of the resulting sequence of input moves computed from the OCP at a sampling time

Multistep Feedback (MF) MPC



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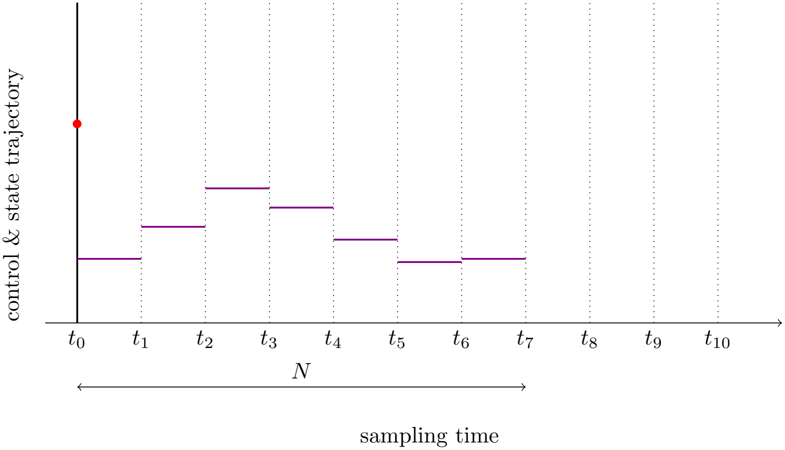

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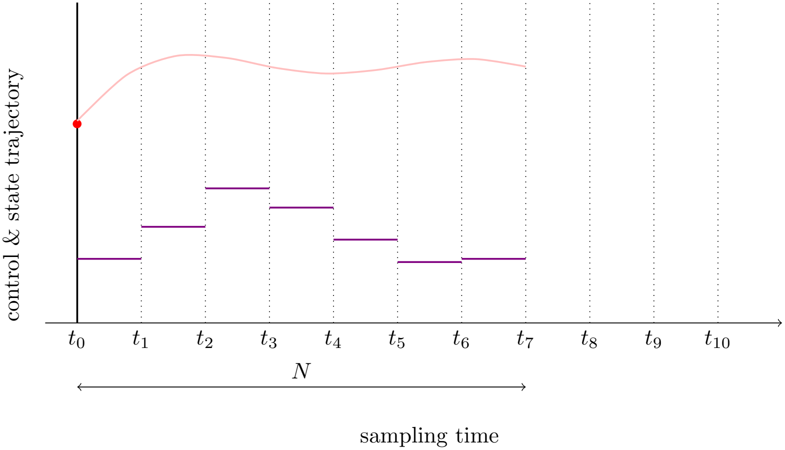

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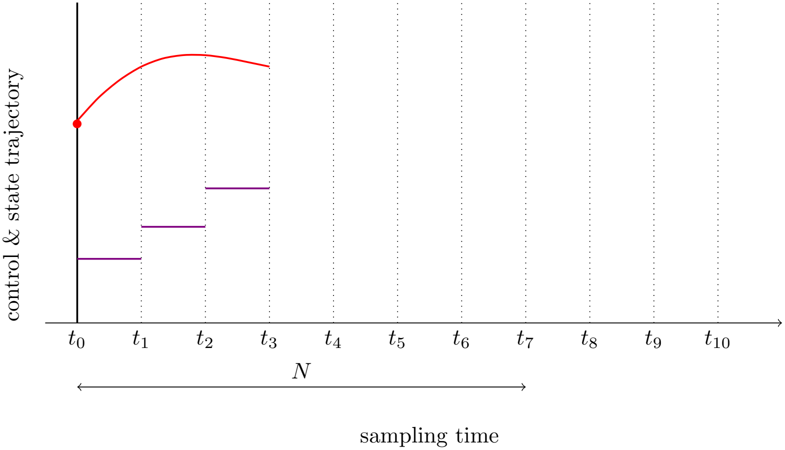

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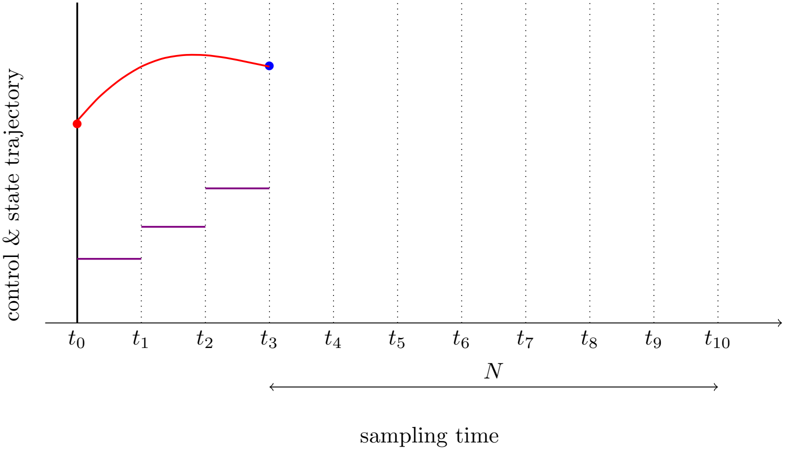

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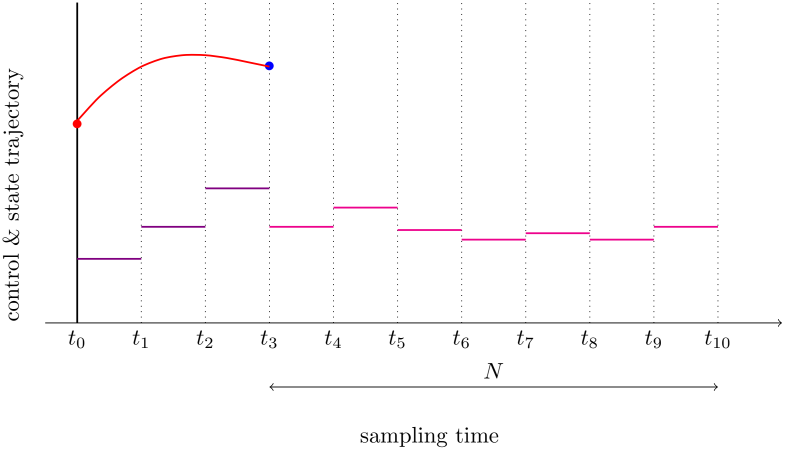

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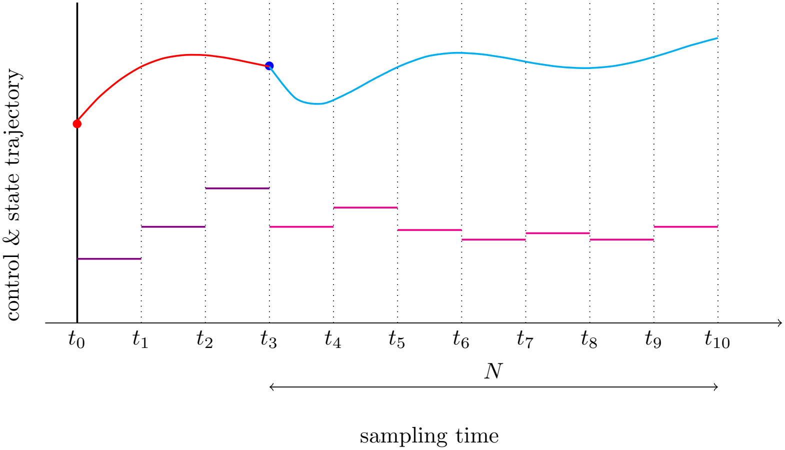

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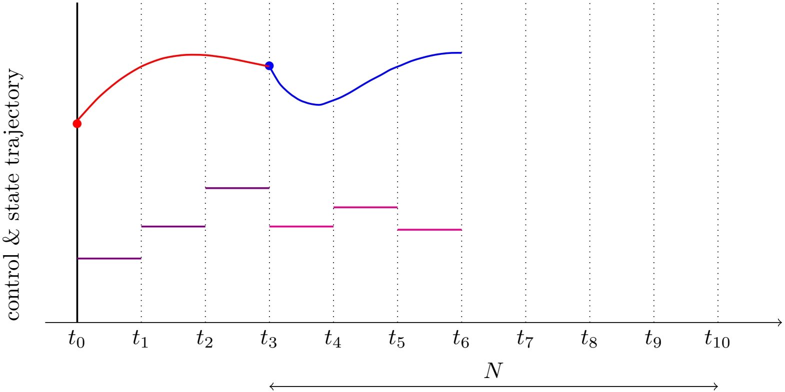
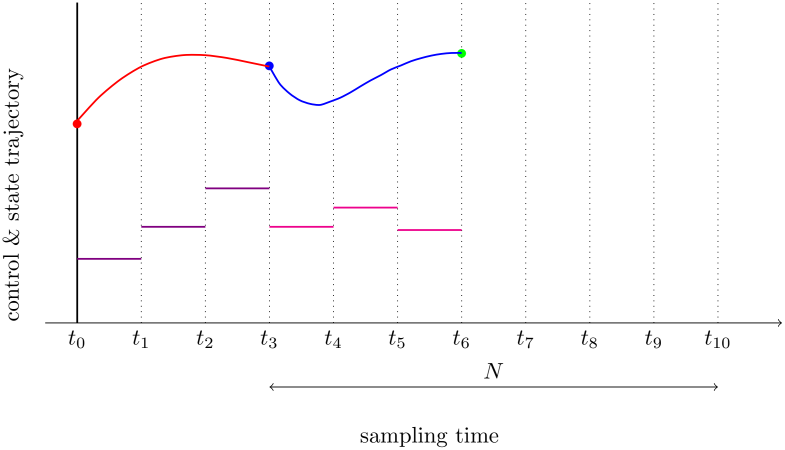


sampling time

Multistep Feedback (MF) MPC



Multistep Feedback (MF) MPC

- the multistep predictions and control moves would be based on old information
- adverse effects of unmeasured disturbances $\Longrightarrow$ reduced robustness
- How can robustness be improved?
- Attempt to use sensitivities!

Sensitivity-based Multistep Feedback (SBMF) MPC

- a method based on parametric sensitivity analysis of a nonlinear programming problem (NLP) to calculate approximations of optimal solutions of problems depending on neighboring parameter

- Consider the NLP $\mathcal{P}_N(p)$

$$\min_z f(z,p) \quad \text{s.t.} \quad g(z,p) = 0 \qquad h(z,p) \geq 0$$

where p is a given parameter and $z(p)$ is the optimization variable.

Sensitivity-based Multistep Feedback (SBMF) MPC

- NLP Sensitivity Theorem *(Fiacco 1976)*
  Under certain assumptions, for p in the some neighborhood of nominal p_0, it holds that

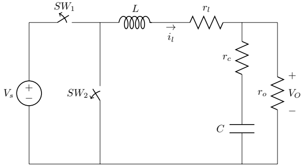
$$\begin{bmatrix} \nabla^2_{zz}\mathcal{L}(z^*,\mu^*,p_0) & \nabla_z C_{\mathcal{A}}(z^*,p_0)^\top \\ \nabla_z C_{\mathcal{A}}(z^*,p_0) & 0 \end{bmatrix} \cdot \underbrace{\begin{bmatrix} \dfrac{\partial z}{\partial p}(p_0) \\ \dfrac{\partial \mu_{\mathcal{A}}}{\partial p}(p_0) \end{bmatrix}}_{\text{sensitivity matrix}} = - \begin{bmatrix} \nabla^2_{zp}\mathcal{L}(z^*,\mu^*,p_0)^\top \\ \nabla_p C_{\mathcal{A}}(z^*,p_0)^\top \end{bmatrix}$$

- sensitivities can be used to obtain a linear approximation for the solution of a perturbed OCP

$$\underbrace{u_0^*(x_0)}_{\text{sol of pertubed problem}} = \underbrace{u_0^*(\tilde{x_0})}_{\text{sol of nominal problem}} + \underbrace{\frac{\partial u_0^*}{\partial x_0^*}}_{\text{sensitivity}} \underbrace{(x_0 - \tilde{x_0})}_{\text{perturbation}}$$

An Example Problem: DC-DC Converter

Controller Design



- synchronous stepdown converter - a switching electronic circuit
- consider the LQ problem defined by the cost

$$J_c = x(T)^\top P_c x(T) + \int_0^T \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^\top \begin{bmatrix} Q_c & 0 \\ 0 & R_c \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt$$

- continuous-time model

$$\dot{x}(t) = \begin{cases} A_c x(t) + b_c, & kT_s \leq t \leq (k + d(t))T_s \\ A_c x(t), & (k + d(t))T_s \leq t \leq (k+1)T_s \end{cases}$$

An Example Problem: DC-DC Converter

MPC Problem

the core optimization problem solved at each time instant is

$$\min_{x,u} \quad x_N^\top P x_N + \sum_{k=0}^{N-1} \begin{bmatrix} x_k \\ u_k \end{bmatrix}^\top \begin{bmatrix} Q & M \\ M^\top & R \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}$$

$$\begin{array}{rrcll} \text{s.t.} & x_0 & = & [\alpha, \beta]^\top & \\ & x_{j+1} & = & Ax_j + bu_j & j = 0, 1, \ldots, N-1 \\ & [0,0]^\top & \leq & x_{j+1} \leq [i_{l\mathsf{max}}, V_s]^\top & j = 0, 1, \ldots, N-1 \\ & 0 & \leq & u_j \leq 1 & j = 0, 1, \ldots, N-1. \end{array}$$

An Example Problem: DC-DC Converter

Incorporating Sensitivities

- first apply obtained u_0^* and then we apply corrected optimal controls $\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_{m-1}$.
- instead of optimizing again at time instants $t_1, t_2, \ldots, t_{m-1}$ (as in single-step MPC), we calculate

$$\frac{\partial u_1^*}{\partial x_1^*}, \frac{\partial u_2^*}{\partial x_2^*}, \ldots, \frac{\partial u_{m-1}^*}{\partial x_{m-1}^*}$$

- SBMF is given by the update rule

$$\tilde{u}_i \quad = \quad u_i^* + \underbrace{\frac{\partial u_i^*}{\partial x_i^*}\left(x_i^{(m)} - x_i^*\right)}_{\text{the update/correction}} \qquad i = 1, \ldots, m-1.$$

- at the time instant t_m, we solve the OCP again

An Example Problem: DC-DC Converter

Incorporating Sensitivities

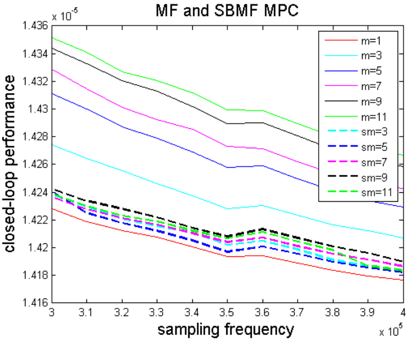
- the sensitivities $\frac{\partial u_1^*}{\partial x_1^*}, \frac{\partial u_2^*}{\partial x_2^*}, \ldots, \frac{\partial u_{m-1}^*}{\partial x_{m-1}^*}$ are computed via

$$\begin{bmatrix} \nabla_{zz}^2 \mathcal{L}_i & \nabla_z C_{\mathcal{A}^i}^\top \\ \nabla_z C_{\mathcal{A}^i} & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial z^i}{\partial x_i}(x_i^*) \\ \frac{\partial \mu_{\mathcal{A}^i}}{\partial x_i}(x_i^*) \end{bmatrix} = - \begin{bmatrix} \nabla_{zp}^2 \mathcal{L}_i^\top \\ \nabla_p C_{\mathcal{A}^i}^\top \end{bmatrix}$$

  for $i = 0, \ldots, m-1$

- solving a sequence of linear systems corresponding to OCPs $\mathcal{P}_{N-i}(x_i^*)$ of decreasing horizon and adjusting parametric initial state value

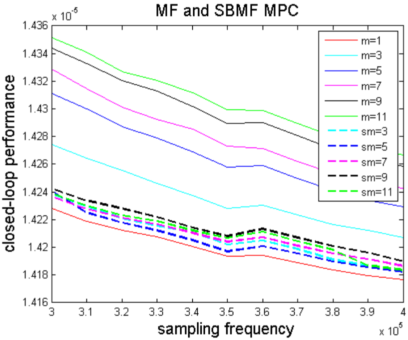
Closed-loop Performance



- $$J_{\mathsf{cl}} = x_{N_T}^\top P x_{N_T} + \sum_{k=0}^{N_T-1} \begin{bmatrix} x_k \\ \mu_k \end{bmatrix}^\top \begin{bmatrix} Q & M \\ M^\top & R \end{bmatrix} \begin{bmatrix} x_k \\ \mu_k \end{bmatrix}$$

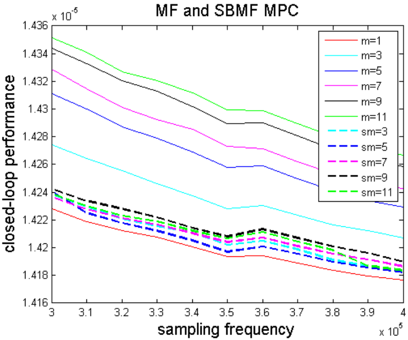
  for discrete simulation time N_T

Closed-loop Performance



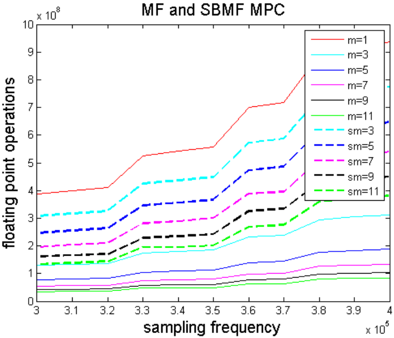
- the performance improves along increasing sampling frequency f_s, suffers by increasing multistep m for MF and improves with SBMF MPC

Closed-loop Performance



- the magnitude of perturbation affects the results/trends

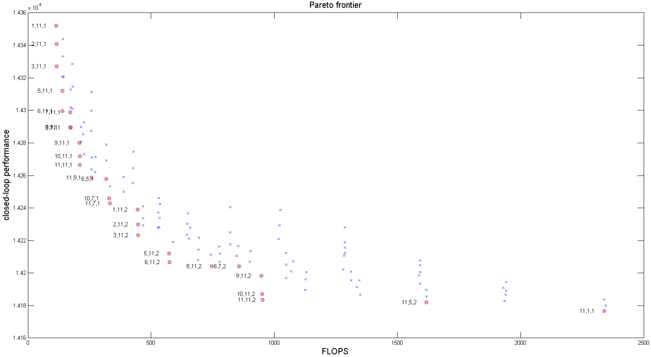
Floating-point Operations (FLOPs)



- for fixed f_s and simulation time length, FLOPs for MF MPC is $\sim \mathcal{O}(N^3)/m$
- for SBMF, some reasonable amount of FLOPs is the cost of better performance

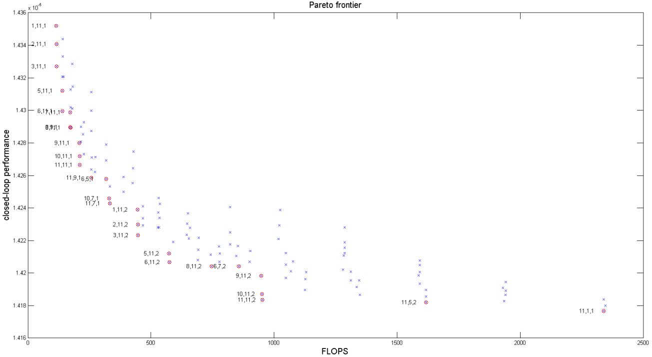
Pareto Efficiency Analysis

- Pareto efficiency is a state of tuning of design parameters in which it is impossible to be better off in a certain criterion without making at being worse off in another criterion



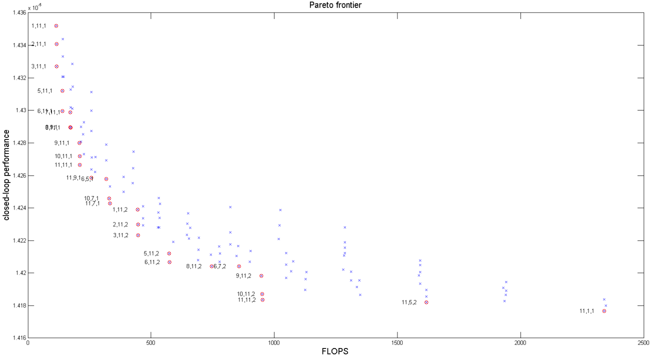
Pareto Efficiency Analysis

- given a set of feasible options, we obtain the Pareto frontier – set of choices that are Pareto efficient



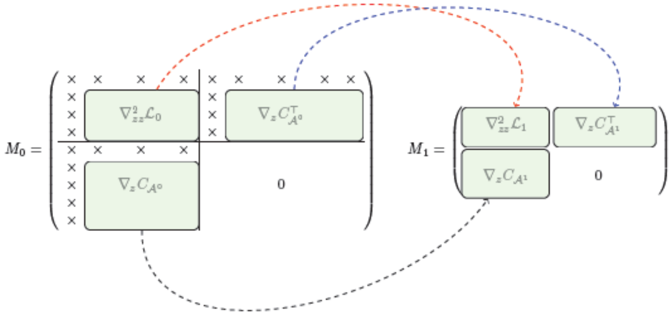
Pareto Efficiency Analysis

- by restricting attention to the set of choices that are Pareto efficient, a control algorithm-hardware designer can analyze and weigh trade-offs within this set



Exploiting Matrix Structures

- The coefficient matrix of the linear system for solving sensitivities for $\mathcal{P}_{N-i}(x_i^*), i = 0, \ldots, m-1$ can be constructed from submatrices of the coefficient matrix solved for $P_N(x_0^*)$
- these are information we get for free!



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