HAMILTON-JACOBI EQUATIONS : APPROXIMATIONS, NUMERICAL ANALYSIS AND APPLICATIONS

CIME Courses-Cetraro
August 29-September 3 2011

COURSES

(1) Models of mean field, Hamilton-Jacobi-Bellman Equations and numerical methods. Yves Achdou.
(2) First-order Hamilton-Jacobi Equations and Applications. Guy Barles.
(3) Basic properties of viscosity solutionsand some aspects of weak KAM theory. Hitoshi Ishii.
(4) Idempotent/Tropical Analysis and the Hamilton-Jacobi-Bellman Equations. Grigory L. Litvinov.
(5) Homogenization and Approximation for Hamilton-Jacobi equations. Panagiotis E. Souganidis.

The courses [2] and [3] will be mainly devoted to an introduction to the theory. The course [4] will introduce idempotent mathematics to evidence analogs between max (or min) non linearity in partial differential equations. The courses [1] and [5] will be more advanced (so they will follow in the time schedule [2] and [3]) and they will be devoted to applications and numerical results.
The final version of the texts of the lectures will be published in the Springer Lecture Notes in Mathematics, CIME Subseries. A preliminar version (slides on line) during the course will be helpful to the understanding.

	August 29	August 30	September 1st	September 2	September 3
9.00 10.00	Barles 1	Ishii 3	Litvinov 4	Achdou 2	Souganidis 4
10.15 11.15	Barles 2	Ishii 4	Litvinov 5	Achdou 3	Souganidis 5
11.15 Coffe Break					
11.45 12.45	Litvinov 1	Achdou 1	Souganidis 1	Barles 5	Ishii 5
13.00 Lunch					
16.45-17.45	Ishii 1	Litvinov 2	Barles 3	Souganidis 2	Achdou 4
18.00-19.00	Ishii 2	Litvinov 3	Barles 4	Souganidis 3	Achdou 5
20.00 Dinner					

1

[1] Models of mean field, Hamilton-Jacobi-Bellman Equations, and numerical methods. Yves Achdou.

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1.1. **CIME course program.** Models of mean field type for the limit of Nash equilibria for stochastic game problems see [3] when the number of players tends to $+\infty$ have recently been studied by J-M. Lasry and P-L. Lions, see [5, 7, 8]. The main assumptions are that all the N players are identical and that each player chooses his optimal strategy in view of a global (partial) information on the game. At the limit a system of two coupled equations is obtained: a forward in time Hamilton-Jacobi-Bellman for a value function and a backward in time Kolmogorov equation for a probability measure. Uniqueness is obtained under some reasonable assumptions. Infinite horizon games will also be considered.
The following points will be discussed:

- some notions on the asymptotic behavior of the Nash equilibria when $N \to \infty$, (a brief and incomplete review of the theory of Lasry and Lions)
- existence for the previously mentioned system of PDEs in the finite horizon case
- uniqueness
- numerical methods for approximating the above mentioned system and numerical analysis, see [1, 2, 4, 5].

References

[1] Y. Achdou and I. Capuzzo Dolcetta. Mean field games: Numerical methods. Technical report, Laboratoire J-L. Lions, University P. et M. Curie, 2009.
[2] Y. Achdou and I. Capuzzo Dolcetta. Mean field games: Numerical methods for the finite horizon problem. in preparation.
[3] A. Bensoussan and J. Frehse. *Regularity results for nonlinear elliptic systems and applications*, volume 151 of *Applied Mathematical Sciences*. Springer-Verlag, Berlin, 2002.
[4] O. Guéant. A reference case for mean field games models. Technical report, CEREMADE, U. Paris Dauphine, 2008.
[5] A. Lachapelle, J. Salomon, and G. Turinici. A monotonic algorithm for a mean field games model in economics. Technical report, CEREMADE, U. Paris Dauphine, 2009.
[6] J.-M. Lasry and P.-L. Lions. Jeux à champ moyen. I. Le cas stationnaire. *C. R. Math. Acad. Sci. Paris*, 343(9):619–625, 2006.
[7] J.-M. Lasry and P.-L. Lions. Jeux à champ moyen. II. Horizon fini et contrôle optimal. *C. R. Math. Acad. Sci. Paris*, 343(10):679–684, 2006.
[8] J.-M. Lasry and P.-L. Lions. Mean field games. *Jpn. J. Math.*, 2(1):229–260, 2007.

2

[2] First-order Hamilton-Jacobi Equations and Applications. Guy Barles.

Guy Barles, Laboratoire de Mathématiques et Physique Théorique CNRS UMR 6083 (Tours), Fédération Denis Poisson, Université François Rabelais Tours, Parc de Grandmont 37200 TOURS, France.

2.1. **CIME course program.** The topic will be mainly devoted to first order Hamilton-Jacobi Equation : notion of viscosity solutions, main properties, comparison theorems, stability, Lipschitz regularity of solutions and further regularity, lower bounds on the gradient, applications to dislocation equations.

References

[1] Bardi, Martino ; Capuzzo-Dolcetta, Italo . *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations. With appendices by Maurizio Falcone and Pierpaolo Soravia.* Systems & Control: Foundations & Applications. Birkhuser Boston, Inc., Boston, MA, 1997.

[2] Barles, Guy Solutions de viscosité des équations de Hamilton-Jacobi Vol.17 Springer-Verlag, Paris, 1994.

[3] Barles, Guy and Cardaliaguet, Pierre and Ley, Olivier and Monneau, Régis *Global existence results and uniqueness for dislocation equations.*, 40 ,44–69 ,of *SIAM J. Math. Anal.*, 2008.

[4] Crandall, Michael G. and Lions, Pierre-Louis Viscosity solutions of Hamilton-Jacobi equations *Trans. Amer. Math. Soc.*, 277, 1983.

[5] Ley, Olivier Lower-bound gradient estimates for first-order Hamilton-Jacobi equations and applications to the regularity of propagating fronts *Adv. Differential Equations*, 6(5) :547–576, 2001.

[3] Basic properties of viscosity solutions and some aspects of weak KAM theory. Hitoshi Ishii.

Hitoshi Ishii, Department of Mathematics, Waseda University, Nishi-Waseda, Shinjuku, Tokyo, 169- 8050 Japan, hitoshi.ishii@waseda.jp

3.1. **CIME course program.** Basic properties of viscosity solutions, some aspects of weak KAM theory, as well as comparison theorems, regularity results and asymptotic analysis for first-order and second-order nonlinear partial differential equations.

A reference to this course could be

References

[1] M. G. Crandall, H.Ishii, P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations. Bull. Amer. Math. Soc. (N.S.) 27 (1992), no. 1, 1–67.

[2] Y. Fujita, H. Ishii, P. Loreti, Asymptotic solutions of viscous Hamilton-Jacobi equations with Ornstein-Uhlenbeck operator. Comm. Partial Differential Equations 31 (2006), no. 4-6, 827–848.

[3] H. Ishii, H. Mitake, Representation formulas for solutions of Hamilton-Jacobi equations with convex Hamiltonians. Indiana Univ. Math. J. 56 (2007), no. 5, 2159–2183

4

[4] Idempotent/Tropical Analysis and the Hamilton-Jacobi-Bellman Equation. Grigory L. Litvinov.

Grigory L. Litvinov, Independent University of Moscow and the Russian-French Laboratory "J.-V. Poncelet" Bol'shoi Vlasievskii per., 11 Moscow 119002, Russia. islc@dol.ru.

4.1. **CIME course program.** The Maslov dequantization and tropical mathematics. Idempotent mathematics and the idempotent correspondence principle.

The Hamilton-Jacobi equation as a result of the Maslov dequantization for the Schroedinger equation. Linearity of the Hamilton-Jacobi-Bellman equation over tropical algebras. The Maslov superposition principle. Idempotent analysis and viscosity solutions. Bellman equations and optimization. Matrix Bellman equations. Numerical solutions and universal algorithms. Idempotent interval analysis.

References

[1] G.L. Litvinov, The Maslov dequantization, idempotent and tropical mathematics: A brief introduction. Journal of Mathematical Sciences, v. 140, 3, 2007, p. 426-444. E-print arXiv:math.GM/0507014 (http://arXiv.org).

[2] G.L. Litvinov and V.P. Maslov, The Correspondence Principle for Idempotent Calculus and Some Computer Applications; Preprint IHES, Bures-sur-Yvette, 1995; the same in: Idempotency/J. Gunawardena (Editor), Cambridge University Press, Cambridge, 1998 (ISBN 0-521-55344-X), p. 420-443. E-print arXiv:math.GM/0101021 (http://ArXiv.org).

[3] G.L. Litvinov and A.N. Sobolevskii, Idempotent interval analysis and optimization problems. Reliable Computing, v. 7, 5, 2001, p. 353-377. E-print arXiv:math.SC/0101080 (http://ArXiv.org).

[4] G.L. Litvinov and V.P. Maslov, Eds., Idempotent Mathematics and Mathematical Physics. AMS Contemporary Mathematics, Vol. 377, 2005.

[5] Homogenization and Approximation for Hamilton-Jacobi equations..
Panagiotis E. Souganidis.

Panagiotis E. Souganidis, Department of Mathematics, The University of Chicago, 5734 S. University Avenue, Chicago, IL 60637

5.1. **CIME course program.** First- and second-order Hamilton-Jacobi equations, Homogenization of fully nonlinear equations in random media, Approximations of viscosity solutions and rates of convergence.

References

[1] G. B. Barles, P. E. Souganidis, Convergence of Approximation schemes for fully nonlinear second order equations, Asymptotic Anal. 4 (1991), no. 3, 271–283.
[2] L. A. Caffarelli, P. E. Souganidis, A rate of convergence for monotone finite difference approximations to fully nonlinear, uniformly elliptic PDEs. Comm. Pure Appl. Math. 61 (2008), no. 1, 1–17.
[3] L. A. Caffarelli, P. E. Souganidis, L. Wang, Homogenization of fully nonlinear, uniformly elliptic and parabolic partial differential equations in stationary ergodic media, Comm. Pure Appl. math. 58 (2005), no. 3, 319–361.
[4] P.-L. Lions, P. E. Souganidis, Homogenization of degenerate second-order PDE in periodic and almost periodic environments and applications Ann. Inst. H. Poincare Anal. Non Lineaire 22 (2005), no. 5, 667–677.
[5] P.-L. Lions, P. E. Souganidis, Homogenization of "viscous" Hamilton-Jacobi equations in stationary ergodic media, Comm. Partial Differential Eq. 30 (2005), no. 1-2, 335–375.
[6] P. E. Souganidis, Stochastic Homogenization of Hamilton-Jacobi equations, Asymptotic Anal.20 (1999), no. 1 , 1–11.