Exponential mapping for the sub-Riemannian problem on the Engel group

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Summary

The left–invariant sub-Riemannian problem on the Engel group is considered. This problem is very important as nilpotent approximation of nonholonomic systems in four–dimensional space with two–dimensional control (see [1, 2]), for instance of a system which describes movement of mobile trailer robot.
Parameterization of extremal curves by elliptic Jacobi's functions was obtained. Discrete symmetries of Exponential mapping were considered and the corresponding Maxwell sets were constructed. Thus global bound of the cut time (i. e., the time of loss of *global* optimality) was found which gives necessary optimality conditions for extremal curves. The first conjugate time (i. e., the time of loss of *local* optimality) was investigated. It was shown that the function that gives the upper bound of the cut time provides the lower bound of the first conjugate time.

Optimal control problem statement

$$\dot q = \begin{pmatrix}\dot x\\ \dot y\\ \dot z\\ \dot v\end{pmatrix} = u_1\begin{pmatrix}1\\0\\-\frac{y}{2}\\0\end{pmatrix} + u_2\begin{pmatrix}0\\1\\\frac{x}{2}\\\frac{x^2+y^2}{2}\end{pmatrix},\quad q\in\mathbb{R}^4,\quad u\in\mathbb{R}^2, \tag{1}$$

$$q(0)=q_0=(x_0,y_0,z_0,v_0),\quad q(t_1)=q_1=(x_1,y_1,z_1,v_1), \tag{2}$$

$$l=\int_0^{t_1}\sqrt{u_1^2+u_2^2}\,dt\to\min. \tag{3}$$

Since the problem is invariant under left shifts on the Engel group, we can assume that the initial point is identity of the group $q_0=(x_0,y_0,z_0,v_0)=(0,0,0,0)$.

Hamiltonian system

Existence of optimal solutions of problem **(1)**–**(3)** is implied by Filippov's theorem [4]. By Cauchy–Schwarz inequality, it follows that sub-Riemannian length minimization problem **(3)** is equivalent to action minimization problem:

$$\int_0^{t_1}\frac{u_1^2+u_2^2}{2}\,dt\to\min. \tag{4}$$

Pontryagin's maximum principle [3, 4] was applied to the resulting optimal control problem **(1)**, **(2)**, **(4)**. Abnormal extremals were parameterized. Denote vector fields at the controls in the right-hand side of system **(1)**:

$$X_1=(1,0,-\frac{y}{2},0)^T,\qquad X_2=(0,1,\frac{x}{2},\frac{x^2+y^2}{2})^T,$$

and the corresponding linear on fibers of the cotangent bundle T^*M Hamiltonians $h_i(\lambda)=\langle\lambda,X_i(q)\rangle$, $\lambda\in T^*M$, $i=1,2$.
Normal extremals satisfy the Hamiltonian system

$$\dot\lambda=\vec H(\lambda),\qquad \lambda\in T^*M, \tag{5}$$

where $H=\frac{1}{2}\left(h_1^2+h_2^2\right)$.
The normal Hamiltonian system **(5)** is given, in certain natural coordinates, as follows on a level surface $\left\{\lambda\in T^*M \mid H=\frac{1}{2}\right\}$:

$$\dot\theta=c,\qquad \dot c=-\alpha\sin\theta,\qquad \dot\alpha=0,$$
$$\dot q=\cos\theta\,X_1(q)+\sin\theta\,X_2(q),\qquad q(0)=q_0. \tag{6}$$

Parameterization of normal extremal trajectories

The family of all normal extremals is parameterized by points of the phase cylinder of pendulum

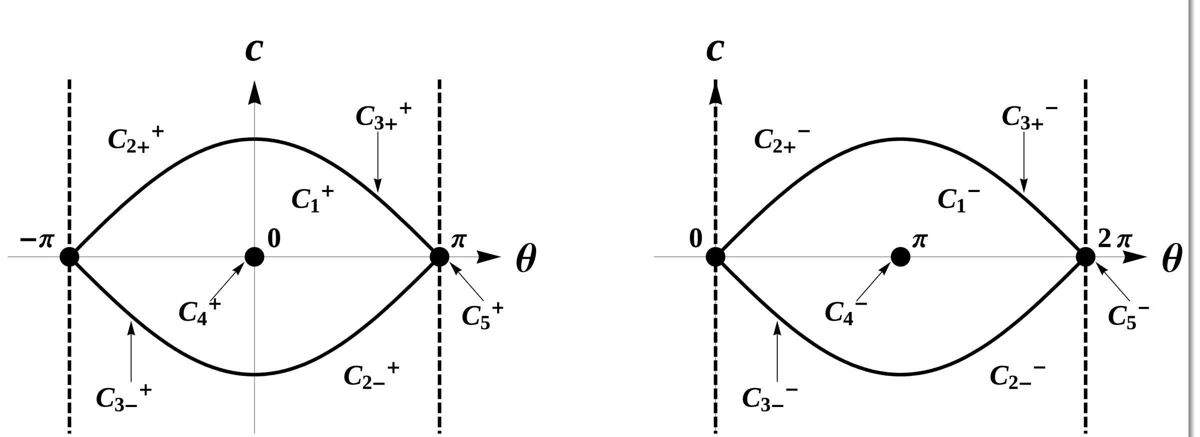
$$C=\left\{\lambda\in T^*_{q_0}M \mid H(\lambda)=\frac{1}{2}\right\}=\left\{(\theta,c,\alpha)\mid\theta\in S^1,\ c,\alpha\in\mathbb{R}\right\},$$

and is given by the exponential mapping

$$\mathrm{Exp}\ :\ N=C\times\mathbb{R}_+\to M,$$
$$\mathrm{Exp}(\lambda,t)=q_t=(x_t,y_t,z_t,v_t).$$

Energy integral of pendulum **(6)** is expressed by $E=\frac{c^2}{2}-\alpha\cos\theta$. The cylinder C has the following stratification corresponding to the particular type of trajectories of the pendulum:

$$C=\cup_{i=1}^7 C_i,\quad C_i\cap C_j=\emptyset,\ i\neq j,\quad \lambda=(\theta,c,\alpha),$$
$$C_1=\{\lambda\in C\mid\alpha\neq0,E\in(-|\alpha|,|\alpha|)\},$$
$$C_2=\{\lambda\in C\mid\alpha\neq0,E\in(|\alpha|,+\infty)\},$$
$$C_3=\{\lambda\in C\mid\alpha\neq0,E=|\alpha|,c\neq0\},$$
$$C_4=\{\lambda\in C\mid\alpha\neq0,E=-|\alpha|\},$$
$$C_5=\{\lambda\in C\mid\alpha\neq0,E=|\alpha|,c=0\},$$
$$C_6=\{\lambda\in C\mid\alpha=0,\ c\neq0\},$$
$$C_7=\{\lambda\in C\mid\alpha=c=0\}.$$



Extremal trajectories were parameterized by elliptic Jacobi's functions for any $\lambda\in C$ in the paper [5]. This parameterization was obtained in natural coordinates (φ,k,α), which rectify the equations of pendulum: $\dot\varphi=1$, $\dot k=0$, $\dot\alpha=0$.

Cut time

In order to investigate the optimality question for discovered extremal trajectories descrete group of symmetries of exponential mapping were considered:

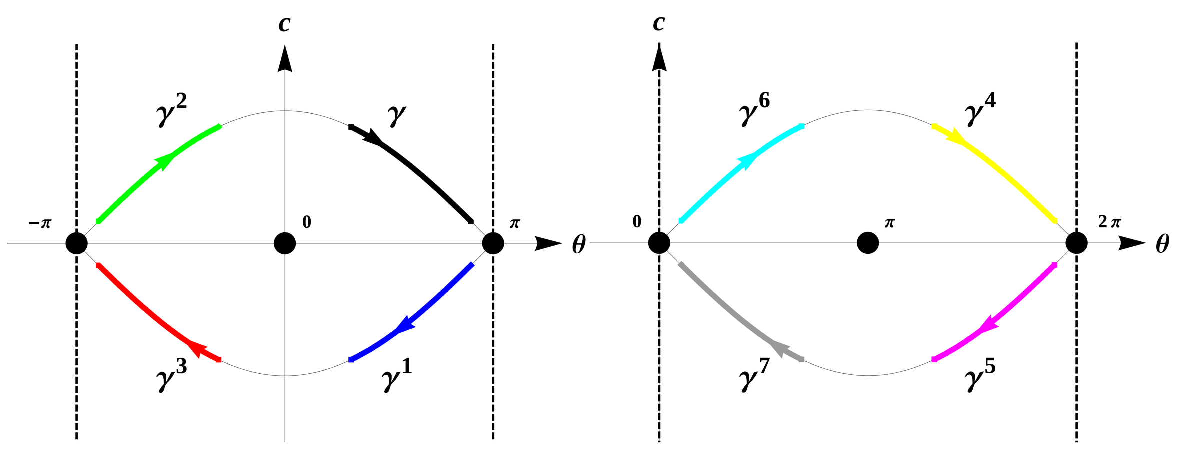


Figure: Action of the symmetries: $\varepsilon^i(\gamma)=\gamma^i$, $i=1\dots7$

Thus the corresponding Maxwell sets were constructed. The point of sub-Riemannian geodesic is called Maxwell point if two different extremal trajectories come to this point at the same time called Maxwell time $t^1_{\mathrm{MAX}}\ :\ C\to(0,+\infty]$:

$$\lambda\in C_1\quad\Rightarrow\quad t^1_{\mathrm{MAX}}=\min(2p^1_z,4K)/\sigma,$$
$$\lambda\in C_2\quad\Rightarrow\quad t^1_{\mathrm{MAX}}=2Kk/\sigma,$$
$$\lambda\in C_6\quad\Rightarrow\quad t^1_{\mathrm{MAX}}=\frac{2\pi}{|c|},$$
$$\lambda\in C_3\cup C_4\cup C_5\cup C_7\quad\Rightarrow\quad t^1_{\mathrm{MAX}}=+\infty.$$

where $\sigma=\sqrt{|\alpha|}$; $K(k)=\int_0^{\frac{\pi}{2}}\frac{dt}{\sqrt{1-k^2\sin^2 t}}$;
$p^1_z(k)\in(K(k),3K(k))$ is the first positive root of the function $f_z(p,k)=\mathrm{dn}\,p\ \mathrm{sn}\,p+(p-2\,\mathrm{E}(p))\,\mathrm{cn}\,p$; $\mathrm{E}(p)=\int_0^p\mathrm{dn}^2\,t\,dt$; $\mathrm{sn}\,p$, $\mathrm{cn}\,p$ and $\mathrm{dn}\,p$ are Jacobi's functions [6].
It is well known that geodesic cannot be optimal after Maxwell point. Thus Maxwell time gives upper bound of the cut time:

$$t_{\mathrm{cut}}(\lambda)=\sup\{t>0\mid\mathrm{Exp}(\lambda,s)\text{ is global optimal for }s\in[0,t]\}.$$

Theorem (1)

For any $\lambda\in C$

$$t_{\mathrm{cut}}(\lambda)\le t^1_{\mathrm{MAX}}(\lambda). \tag{7}$$

The bound of the cut time obtained in the Theorem (1) is sharp for the equilibrium of the pendulum, i.e. the corresponding trajectories are optimal to infinity. Analysis of the global structure of the exponential map shows that found estimate is not exact in general case.

Conjugate time

The local optimality of extremal trajectories was studied. A point $q_t=\mathrm{Exp}(\lambda,t)$ is called *a conjugate point* for q_0 if $\nu=(\lambda,t)$ is a critical point of the exponential mapping and that is why q_t is the corresponding critical value:

$$d_\nu\,\mathrm{Exp}\ :\ T_\nu N\to T_{q_t}M\text{ is degenerate},$$

i. e.,

$$\frac{\partial(x,y,z,v)}{\partial(\theta,c,\alpha,t)}(\nu)=0.$$

Note that t in this case is called *a conjugate time* along extremal trajectory $q_s=\mathrm{Exp}(\lambda,s)$, $s\ge0$.
Due to the strong Legendre condition, for any normal extremal there exists a countable family of conjugate points. Besides, conjugate times are separated from each other. The first conjugate time along the trajectory $\mathrm{Exp}(\lambda,s)$ is denoted by

$$t^1_{\mathrm{conj}}=\min\{t>0\mid t\text{ is a conjugate time along }\mathrm{Exp}(\lambda,s),\ s\ge0\}.$$

The trajectory $\mathrm{Exp}(\lambda,s)$ loses local optimality at the moment $t=t^1_{\mathrm{conj}}(\lambda)$ (see [4]). The following lower bound of the first conjugate time was proved.

Theorem (2)

For any $\lambda\in C$

$$t^1_{\mathrm{conj}}(\lambda)\ge t^1_{\mathrm{MAX}}(\lambda). \tag{8}$$

Using the estimate of cut time, Theorem (1), and the estimate of conjugate time, Theorem (2), the global structure of the exponential map in sub-Riemannian problem on the Engel group was described. So this problem was reduced to solving the system of algebraic equations.

System of algebraic equations

In order to compute the optimal trajectory for a given terminal point (x_1,y_1,z_1,v_1), the following system of algebraic equations should be solved:

$$\begin{cases}x(u_1,u_2,k,\alpha)=x_1,\\ y(u_1,u_2,k,\alpha)=y_1,\\ z(u_1,u_2,k,\alpha)=z_1,\\ v(u_1,u_2,k,\alpha)=v_1.\end{cases} \tag{9}$$

Using one symmetry (dilations) the system (9) was reduced to the system with three algebraic equations in three unknowns variables:

$$Y(u_1,u_2,k)=Y_1,\quad Z(u_1,u_2,k)=Z_1,\quad V(u_1,u_2,k)=V_1, \tag{10}$$

where $Y=\frac{y}{x}$, $Z=\frac{z}{x^2}$, $V=\frac{v}{x^3}$.
Upper bound of cut time gives decomposition of the preimage $C=\cup_{i=1}^8 D_i$ of exponential map Exp into subdomains D_i.
The image of the exponential mapping was decomposed into subdomains respectively:

$$M=\cup_{i=1}^4 M_i, \tag{11}$$
$$M_1=\{(x,y,z,v)\in\mathbb{R}^4\mid x>0,z>0\}, \tag{12}$$
$$M_2=\{(x,y,z,v)\in\mathbb{R}^4\mid x<0,z<0\}, \tag{13}$$
$$M_3=\{(x,y,z,v)\in\mathbb{R}^4\mid x>0,z<0\}, \tag{14}$$
$$M_4=\{(x,y,z,v)\in\mathbb{R}^4\mid x<0,z>0\}. \tag{15}$$

There is a conjecture that restriction $\mathrm{Exp}:D_i\to M_i$, $\mathrm{Exp}:D_{i+4}\to M_i$ of the exponential map for these subdomains is a diffeomorphism, i. e. $\forall q_1\in M_i\quad\exists!(\lambda,t)\in D_i,\mathrm{Exp}(\lambda,t)=q_1$ and $\forall q_1\in M_i\quad\exists!(\lambda,t)\in D_{i+4},\mathrm{Exp}(\lambda,t)=q_1,i\in\{1,\dots,4\}$.

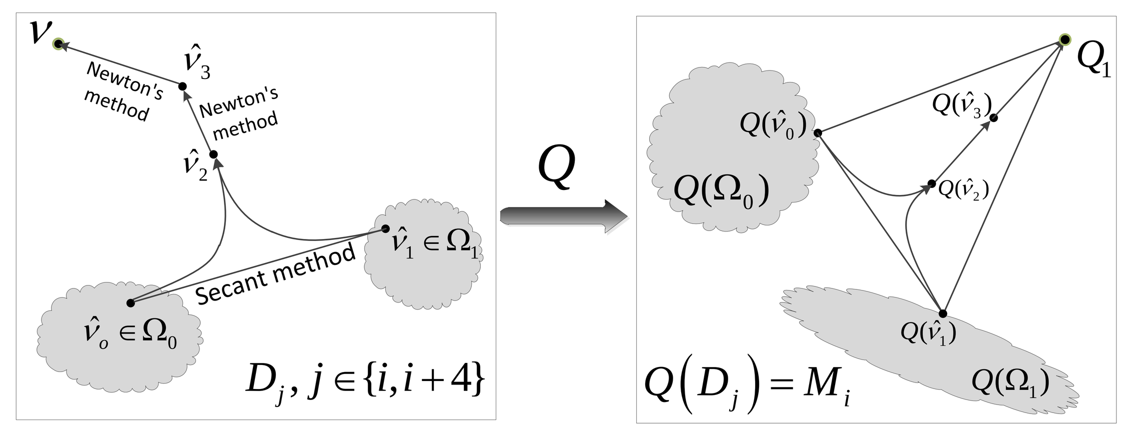


Figure: Hybrid method for solving system of algebraic equations (10)

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Conclusion

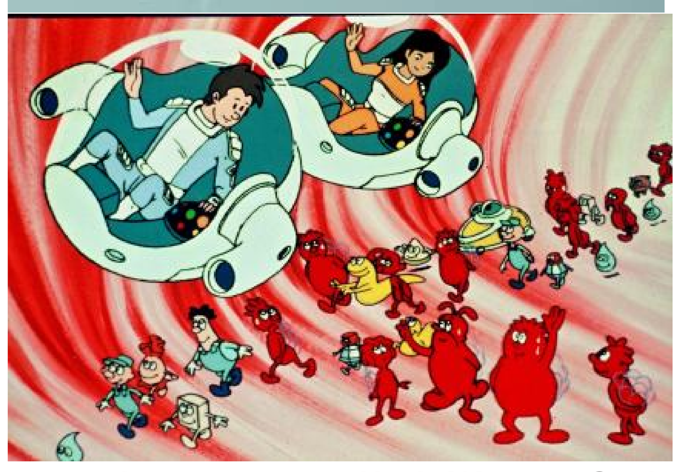
On the basis of these results, software for numerical computation of a global solution to the sub-Riemannian problem on a group of Engel was developed. So solution of the path-planning problem for mobile trailer robot via nilpotent approximation will be developed (this work is in progress).
The method for estimating a conjugate time used in this work was successfully applied earlier to Euler's elastic problem [7] and sub-Riemannian problem on the group of rototranslations [8]. There is no doubt that this method is also valid for nilpotent sub-Riemannian problem with the growth vector (2,3,5) [9, 10, 11, 12]. The method can be used for other invariant sub-Riemannian problems on Lie groups of low-dimensional integrable in non-elementary functions. The first natural step in this direction is investigation of invariant sub-Riemannian problem on 3D Lie groups which are classified by A.A. Agrachev and D.Barilari [13].

Influence of boundary on the motility of micro-swimmers

Laetitia Giraldi*

Motivations

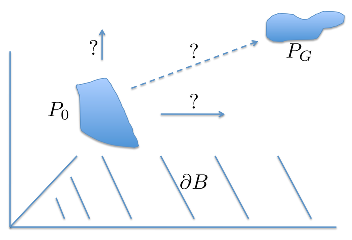
- Self-propulsion at micro-scales?
- Applications on fertility, on human diagnosis and therapy...



- Physicians and biologists noticed that micro-swimmers as Spermatozoid are attracted by the wall. ([H. Winet et al., Reproduction, 1984]).
- **Does the boundary have an effect on the controllability of the swimmer?**

Controllability issues

- Is it possible to control the state of the system?
- Does the boundary impact the controllability of the swimmer?



Model swimmer/fluid

The swimmer is described by the vector (ξ, p) such as :

- ξ is a function which defines the shape of the swimmer.
- $p = (c, R) \in \mathbb{R}^3 \times SO(3)$ parametrizes the swimmer's position.

The swimmer changes its shape $\Longrightarrow \xi(t)$ pushes the fluid.
The fluid reacts, under the Stokes Equation

$$\begin{cases} -\nu \Delta u + \nabla q = f, \\ \text{div} u = 0. \end{cases}$$

Self-propulsion constraints $\Longrightarrow \begin{cases} \sum \text{Forces} = 0 \\ \text{Torque} = 0 \end{cases}$

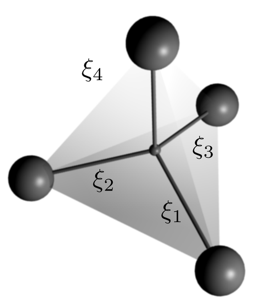
$$\Longleftrightarrow \begin{cases} \int_{\partial\Omega} DN_{p,\xi} \left((\partial_p \Phi)\dot{p} + (\partial_\xi \Phi)\dot{\xi} \right) dx_0 = 0 \\ \int_{\partial\Omega} x_0 \times DN_{p,\xi} \left((\partial_p \Phi)\dot{p} + (\partial_\xi \Phi)\dot{\xi} \right) dx_0 = 0. \end{cases}$$

As a result the swimmer moves, under the ODE,

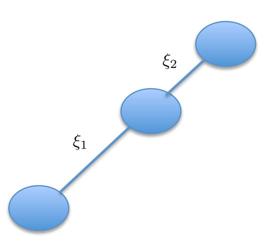
$$\dot{p} = V(p, \xi)\dot{\xi}\,.$$

The swimmers

- The swimmer that we consider consists of n spheres connected by the swimmer's arm.



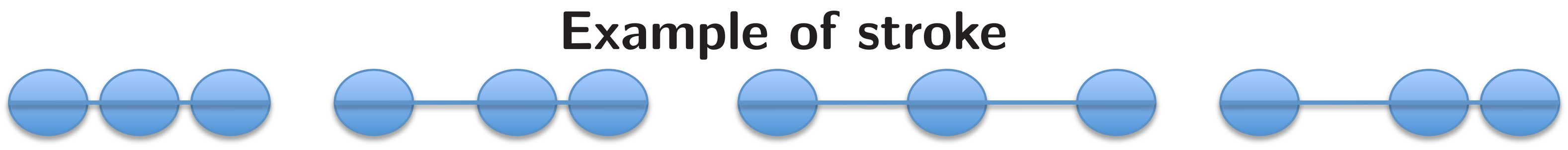
4-sphere swimmer



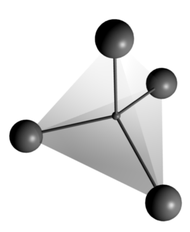
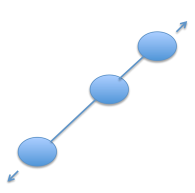
3-sphere swimmer [Golestanian, Najafi 2004]

- The change of the swimmer's shape consists in changing the length of its arms $(\xi_i)_i$.

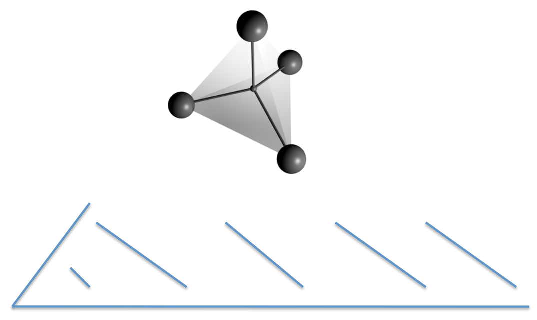
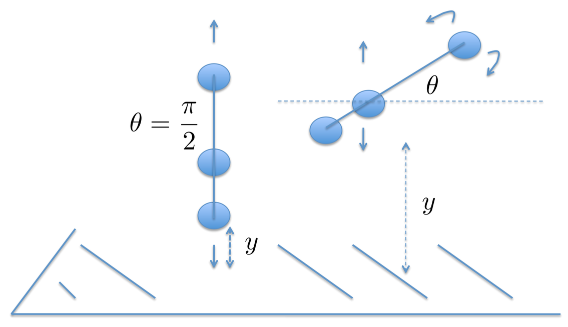
Example of stroke



Controllability's result in $\mathbb{R}^3$ – [Alouges, DeSimone, Heltai, Lefevbre, Merlet]

-  The 4-sphere swimmer is globally controllable on $\mathbb{R}^3$.
-  The 3-sphere swimmer is globally controllable on $\mathbb{R}$.

Influence of a plane wall - Joint work with F. Alouges

-  The 4-sphere swimmer is controllable on an dense open set.
-  For almost (x_0, y_0, θ_0) such that $\theta_0 \neq \frac{\pi}{2}$, the 3-sphere swimmer is locally controllable on (x_0, y_0, θ_0). If $\theta_0 = \frac{\pi}{2}$ then it moves along a vertical line.

Outline of the proofs

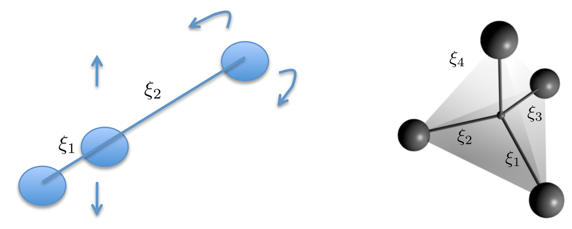
The proofs are based on the study of the subspace $\text{Lie}_{(p,\xi)}((\mathbf{V}_i)_{i=1..M})$, where $(\mathbf{V}_i)_i$ are the vector fields of the motion equation,

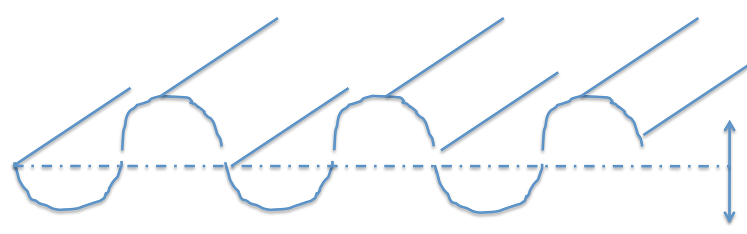
$$\dot{p} = \sum_{i=1}^{M} \mathbf{V}_i(p, \xi)\dot{\xi}\,.$$

- By using the limit and the case without wall
- The orbit with a 3 dimensional Lie space (if $\theta_0 = \frac{\pi}{2}$).
  - By symmetry.
- The others such that the dimension is equal to 5.
  - By using an integral representation of the solution of the Stokes problem, we get an expansion of the Neumann-To-Dirichlet map for large arm and small spheres.
  - Calculation of the Lie brackets and application of Chow Theorem.
  - Application of the Nagano Theorem.

Rough no slip wall - Work in Progress with D. Gérard-Varet

- The 4-sphere remain controllable on an dense open set.
- The dimension of the reachable set of the 3-sphere is greater than or equal to 5.





Outline of the proof

- The Green function of the Stokes problem is implicitly defined.
- Analyticity of the Green Function.
- Analyticity of the Neumann-to-Dirichlet map.
- Expansion of the Neumann-to-Dirichlet map for small ϵ.
- By using the limit of the family of vector-fields which defines the equation of motion,
  - when the altitude of the swimmer is large
  - when the parameter ϵ is small
- Application of the preceding results.

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Extension of Chronological Calculus for Dynamical Systems on Manifolds

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Abstract:

We present an extension of Chronological Calculus to the case of infinite-dimensional C^m-smooth manifolds. The original Chronological Calculus was developed by Agrachev and Gamkrelidze for the study of dynamical systems on C^∞-smooth finite-dimensional manifolds. The extension of this calculus allows for the study of control systems with merely measurable controls and may be applied to C^m-smooth manifolds modeled over Banach spaces. We apply our extension to establish a formula of Mauhart and Michor for the generation of Lie brackets of vector fields and we present a proof of the Chow-Rashevskii theorem on C^m-smooth manifolds modeled over Banach spaces.

Classical Chronological Calculus:

The $C^\infty(M)$ algebra: A central object of study in the Chronological Calculus of Agrachev and Gamkrelidze is the algebra $C^\infty(M)$ of C^∞-smooth functions $f{:}M \to \mathbf{R}$. An important observation is that inherently nonlinear objects such as diffeomorphisms of manifolds give rise to inherently linear objects such as automorphisms of this algebra. Indeed, given a diffeomorphism $A{:}M\to M$, one obtains an automorphism $\hat{A} : C^\infty(M)\to C^\infty(M)$ by $\hat{A}(f) = f\circ A$. There are similar correspondences for points in M, for tangent vectors, and for vector fields. These correspondences provide a means to study many of the nonlinear objects of control theory in a setting where they behave linearly.

The Whitney Topology: Agrachev and Gamkrelidze place a topology on $C^\infty(M)$ in which $f_n \to f$ if and only for any compact subset K of M, one has the uniform convergence over K of f_n and derivatives of all orders *to f*. The precise meaning of this statement can be formulated through the Whitney embedding theorem. Equipped with this topology, $C^\infty(M)$ has the structure of a Fréchet space and the correspondences described above lead to the study of nonlinear objects as linear operators on this space.

Challenges: The classic chronological calculus is unable to handle control problems in which the dynamics are merely C^m-smooth, or are merely measurable in time, or which take place on a manifold whose local structure is infinite dimensional. In addition, the use of Fréchet space structure seems to complicate proofs for a number of important results.

Main Results

Extension of Chronological Calculus: Given a C^m-smooth manifold M modeled over a Banach space E, let $C^r(M,E)$ denote the vector space of r-times differentiable functions $f : M\to E$. The principle setting for our extension is the study of families of operators on these vector spaces. In this way, we are able to develop results for C^m-smooth dynamics on manifolds modeled over Banach spaces. For example, the local flow $P_{s,t}$ of a nonautonomous vector field V_t gives rise under appropriate assumptions to a family of operators $C^r(M,E) \to C^r(M,E)$.

Calculus of Little *o*'s: In order to facilitate use of the calculus, we have developed a calculus of remainder terms, so that one is able to refer to a family of operators Q_t as being *differentiable with derivative* V_t whenever one has $Q_{t+h} = Q_t + hV_t + o(h)$. This rule is satisfied, for example, when Q_t is the flow of an autonomous vector field V. We establish the following useful properties for these operators:

1. $o(t^n) + o(t^n) = o(t^n)$
2. $o(t^n)\circ o(t^m) = o(t^{n+m})$
3. For vector fields V_t and W_t with locally bounded derivatives, $V_t \circ o(t^n) \circ W_t = o(t^n)$
4. If P_t and Q_t are families of operators arising from flows of vector fields, $P_t \circ o(t^n) \circ Q_t = o(t^n)$

These properties lead to simplified proofs of important results such as the bracket formula of Mauhart and Michor.

Product Rule for Composition of Operators: We say that a family of operators is *differentiable at t with derivative* A_t if $P_{t+h} = P_t + h\,A_t + o(h)$. Using the above properties, one may check that if P_t and Q_t are differentiable at t with derivatives A_t and B_t, respectively, then the composition $P_t \circ Q_t$ is differentiable with derivative $A_t \circ Q_t + P_t \circ B_t$.

Flows of Perturbed Vector Fields: Given vector fields V_t and W_t we derive a formula for the flow of their sum as a correction to the flow of V_t. This is done in a general setting which allows the study of perturbations to nonautomomous C^m-smooth vector fields on Banach manifolds which are merely measurable in time.

Bracket Formula: Mauhart and Michor define a *bracket of flows* as $[P_t, Q_t] = P_t \circ Q_t \circ P_t^{-1} \circ Q_t^{-1}$. The calculus of remainder terms gives us an algebraic proof of the following formula of Mauhart and Michor:

$$B(P_t^1, P_t^2, \ldots, P_t^k) = Id + t^k B(X_1, X_2, \ldots, X_k) + o(t^k)$$

where B is a bracket expression.

Absolute Continuity: Given a family B_t of operators, we define its integral in a weak sense through its action on functions – a definition similar to the Dunford-Pettis integral of functional analysis. This in turn provides a definition of a *weak* or *distributional* derivative which is appropriate for an absolutely continuous family of operators. In particular, we say that A_t is absolutely continuous if

$$A_t = A_{t_0} + \int_{t_0}^{t} B_s\, ds$$

And we then say that A_t has derivative B_t in a *weak* or *distributional* sense. For example, if V_t is a nonautonomous vector field which is measurable in time then the flow P_t satisfies

$$P_t = P_{t_0} + \int_{t_0}^{t} P_s \circ V_s\, ds$$

We are able to prove that the composition of such operators is again absolutely continuous and that

$$\int_{t_0}^{t_1} \tfrac{d}{dt}(A_t \circ B_t)\,dt = \int_{t_0}^{t_1} \left(\frac{dA_t}{dt} \circ B_t + A_t \circ \frac{dB_t}{dt} \right) dt$$

Chow-Rashevskii Theorem: We apply the bracket formula of Mauhart and Michor, along with some nonsmooth analysis for manifolds, to prove a variant of the Chow-Rashevskii Theorem on Banach Manifolds. In particular, we prove that if M is modeled over a smooth Banach space, then a smooth affine control system is globally approximately controllable when the Lie algebra of the associated vector fields spans T_qM for any q.

Motion Planning Problem for some Control Systems Applied in Robotics

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Summary

The classic objects of study in robotics are mathematical models of wheeled mobile robots and robots-manipulators. In general, such systems are described by nonlinear nonholonomic control system linear with respect to control $\dot{q} = \sum_{i=1}^{n} u_i(t) X_i(q)$, where the state space $Q \ni q$ is a connected smooth manifold, the controls $(u_1, \dots, u_n) \in \mathbb{R}^n$ are measurable and locally bounded, and $X_1, \dots, X_n$ are smooth vector fields (see [1]). An interesting case occurs when the dimension of the state space exceeds the dimension of control $\dim Q > n > 1$. In generic case the minimal dimension of control $n = 2$ generates a completely controllable system which can reach any desired configuration from any initial configuration. A two-point boundary value problem for such systems is studied. The problem also known as the motion planning problem. The aim is to find controls $(u_1(t), u_2(t))$ which transfer the system from any given initial state $q^0 \in Q$ to any given terminal state $q^1 \in Q$: $q(0) = q^0, \quad q(T) = q^1$. A method of approximate solution based on the nilpotent approximation is used. The general method is concretized for solving the motion planning problem for five-dimensional systems with two-dimensional control: $\dot{q} = u_1(t) X_1(q) + u_2(t) X_2(q), \quad \dim(Q) = 5, \quad \rho(q(T), q^1) < \epsilon$, where ρ is a distance on manifold Q. Specific systems of the type under consideration is the kinematic model of mobile robot with two trailers and the ball rolling on a plane without slipping or twisting.

Statement of the problem

We consider the following motion planning problem

$$\dot{q} = u_1(t) X_1(q) + u_2(t) X_2(q), \quad (1)$$
$$q(0) = q^0, \; x(T) = q^1, \quad (2)$$

where the state space $Q \ni q$ is a connected five-dimensional smooth manifold, control takes values on a two-dimensional plane $(u_1, u_2) \in \mathbb{R}^2$, and the smooth vector fields X_1, X_2 satisfy Lie Algebra Rank condition (LARC) [2] on the manifold Q (i.e. system (1) is completely controllable). Nowadays there are no explicit methods to solve (1)-(2) in general case. Satisfactory solution exists only for certain special classes of systems. However, such problems arise in engineering, where approximate solution is enough, if the error does not exceed a prescribed value. We propose a method to construct the control $(u_1(t), u_2(t))$ that translates system (1) from any initial state q^0 to any terminal state q^1 with any desired precision $\varepsilon > 0$. That is, in such a state $\tilde{q}^1$, that $\rho(\tilde{q}^1, q^1) < \varepsilon$, where ρ is a distance on the manifold Q, for example, if $Q = \mathbb{R}^5$, then $\rho = \sqrt{\sum_{i=1}^{5} (q^1 - \tilde{q}^1)^2}$.

Systems of the form (1) are characterized by the fact that the dimension of the control is less than the dimension of the state space $2 = \dim \mathbb{R}^2 < \dim Q = 5$ but any two points of the state space can be connected by trajectory of the system. In control theory such systems are called completely nonholonomic.

Nonlinear system (1), linear in controls, the number of which is less than dimension of the state space is characterized by different shifts in different directions. The value of displacement in the direction of the fields X_1 and X_2 in a small time t is $O(t)$, in the direction of a commutator $X_3 = [X_1, X_2]$ is $O(t^2)$ in the direction $X_4 = [X_1, X_3]$ and $X_5 = [X_2, X_3]$ is $O(t^3)$. Because of this anisotropy of the state space the control problem for such systems is highly nontrivial.

Controllability

Rashevsky-Chow theorem [2] claims that any two points q^0, $q^1 \in Q$ are reachable from each other if at any point $\tilde{q} \in Q$ linear span of elements of the Lie algebra Lie(X_1, X_2) coincides with the tangent space $T_{\tilde{q}}Q$ *(LARC)*: $\forall \tilde{q} \in Q$ span(Lie(X_1, X_2)) $= T_{\tilde{q}}Q$.

Let us fix $\tilde{q} \in Q$ and define by $\mathrm{L}^s(\tilde{q})$ a vector space generated by the values of Lie brackets X_1, X_2 of length $\leq s$, $s = 1, 2, \dots$ at $\tilde{q}$ (the fields X_i are brackets of length 1):

$$\mathrm{L}^1(\tilde{q}) = \text{span}(X_1(\tilde{q}), X_2(\tilde{q})),$$
$$\mathrm{L}^2(\tilde{q}) = \text{span}(\mathrm{L}^1(\tilde{q}) + [X_1, X_2](\tilde{q})),$$
$$\dots$$
$$\mathrm{L}^s(\tilde{q}) = \text{span}(\mathrm{L}^{s-1}(\tilde{q}) + \{[X_{i_s}, [X_{i_{s-1}}, \dots [X_{i_2}, X_{i_1}] \dots]](\tilde{q}) | i_1, \dots, i_s \in \{1, 2\}\}).$$

LARC ensures that for every $\tilde{q} \in Q$ there exists a smallest integer $r = r(\tilde{q})$ such that $\dim \mathrm{L}^r(\tilde{q}) = 5$. Define *Growth vector* as $(n_1(\tilde{q}), \dots, n_r(\tilde{q}))$, where $n_s(\tilde{q}) = \dim \mathrm{L}^s(\tilde{q})$, $s = 1, \dots, r$. We consider system (1) in a neighborhood of regular points, where growth vector is equal to $(2, 3, 5)$.

Nilpotent approximation

We present a method for finding approximate solutions of the problem (1)–(2) based on nilpotent approximation. Local approximation of a control system by another (simpler) system is often used in control theory. Usually linearization of the control system is used as a local approximation. However, for control systems of the form (1) linearization gives too rough approximation. Since the number of controls less than the dimension of state space, the linearization can not be completely controlled. Natural replacement of the linear approximation in this case gives a nilpotent approximation — the most simple system that preserves the original structure of the control system and therefore controllability (in particular, it remains a growth vector).

We use algorithm of Bellaiche [3] to get nilpotent approximation of original system in a neighborhood of end point q^1 and then we make a change of variables in which the nilpotent approximation has the canonical form:

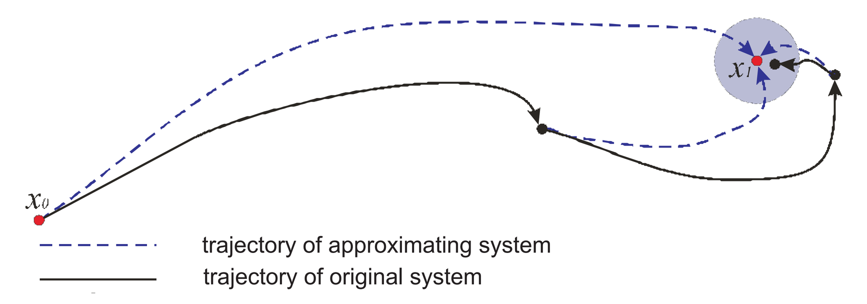
$$\begin{cases} \dot{y}_1 = u_1, \\ \dot{y}_2 = u_2, \\ \dot{y}_3 = \frac{1}{2}(y_1 u_2 - y_2 u_1), \\ \dot{y}_4 = \frac{1}{2}(y_1^2 + y_2^2) u_2, \\ \dot{y}_5 = -\frac{1}{2}(y_1^2 + y_2^2) u_1, \end{cases} \quad y \in \mathbb{R}^5, \quad (3)$$

and boundary conditions are following:

$$Q(0) = Q_1, \qquad Q(T) = 0. \quad (4)$$

Motion Planning Iterative Algorithm

To solve problem (1), (2) we use Iterative algorithm based on the local approximation of the original system by nilpotent system (3), for which the control problem must be solved exactly in each iteration.



So, the problem is to find the control such that corresponding trajectory of system (1) starts from initial state q^0 and ends in a state $\tilde{q}^1$ that satisfied the inequality $\rho(q^1, \tilde{q}^1) < \varepsilon$ for any given $\varepsilon > 0$. To solve the problem we use the following iterative algorithm:

1. building nilpotent approximation at q^1 and computing the change of variables in which nilpotent approximation has form (3);
2. finding a control $(u_1(t), u_2(t))$ in given class of functions that solves problem (3)–(4) for nilpotent system exactly;
3. found control is applyed to the original system, and if the reached state misses the ε-neighborhood of the target state, then the required precision is not achieved, and the step 2 is repeated with the new boundary condition — the state reached by the previous iteration of the algorithm is chosen as new initial state, otherwise calculation stops.

We developed parallel software "MotionPlanning.m" that implements this algorithm as a package for Wolfram Mathematica. It solves the motion planning problem (1), (2) for sufficiently close q^0 and q^1 (it means $\rho(q^0, q^1) < \delta$, where $\delta > 0$ depends on the concrete form of the vector fields in right part of (1) and class of function in which control are to be found). δ must be estimated to establish convergence domain of the algorithm (work in progress). For the present we have a results of numerical experiments.

The software "MotionPlanning.m" solves the motion planning problem in the classes of piece-wise constant controls and optimal controls for nilpotent approximation.

Piece-wise constant control

For any $Q_1 \in \mathbb{R}^5$ exist $(\alpha_i, \beta_i, \gamma_i, \delta_i) \in \mathbb{R}^4$, $i \in \{1, 2\}$ and control

$$u_i = \begin{cases} \alpha_i, \text{ for } t \in [0, \frac{1}{4}], \\ \beta_i, \text{ for } t \in (\frac{1}{4}, \frac{1}{2}], \\ \gamma_i, \text{ for } t \in (\frac{1}{2}, \frac{3}{4}], \\ \delta_i, \text{ for } t \in (\frac{3}{4}, 1], \end{cases}$$

such that $Q(0) = Q_1$, $Q(1) = 0$

- algebraic equations for parameters $(\alpha_i, \beta_i, \gamma_i, \delta_i)$
- nonunique solution
- final fixing of parameters by criterion $\max |u_i| \to \min$

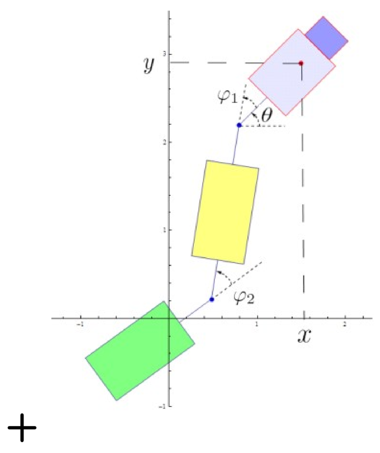
Optimal Control

- (3), (4), $\int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min$
- Nilpotent sub-Riemannian problem with growth vector (2,3,5) (Yu. Sachkov):
  - extremal trajectories, bounds on cut time, global structure of exponential mapping, symmetries, reduction to system of 3 algebraic equations in Jacobian functions of 3 variables
- Optimal synthesis algorithm
- Genetic algorithm for numerical solution of algebraic equations systems

Car with Two Trailers

- state variables
  $\xi = (x, y, \theta, \phi_1, \phi_2)$
  $\xi \in \mathbb{R}^2 \times S^1 \times (S^1 - \{\pi\})^2$
- control system

$$\begin{cases} \dot{x} &= \cos\theta \, u_1, \\ \dot{y} &= \sin\theta \, u_1, \\ \dot{\theta} &= u_2, \\ \dot{\phi}_1 &= -\sin\phi_1 \, u_1 + \\ & +(-1 - \cos\phi_1) \, u_2, \\ \dot{\phi}_2 &= (\sin(\phi_1 - \phi_2) + \sin\phi_1) \, u_1 + \\ & +(\cos(\phi_1 - \phi_2) + \cos\phi_1) u_2. \end{cases}$$



Sphere rolling on a plane

Consider a sphere rolling on a plane whithout slipping or twisting (see [4]). State of the system is described by the contact point between the sphere and the plane and orientation of the sphere in three-dimensional space. One should roll the sphere from any initial contact configuration to any desired configuration. The problem has application in robotics: rotation of a solid body in robot's hand.

Let $(x, y) \in \mathbb{R}^2$ be the contact point of the sphere and the plane.

By $q = (q_0, q_1, q_2, q_3) \in S^3$ denote the unit quaternion (see [5]) representing the rotation of three-dimensional space, which translates the current orientation of the sphere to the initial orientation. The control system described rolling sphere has the following form:

$$\dot{Q} = u_1 X_1(Q) + u_2 X_2(Q),$$

where $X_1(Q) = (1, 0, q_2, q_3, -q_0, -q_1)^T$, $X_2(Q) = (0, 1, -q_1, q_0, q_3, -q_2)^T$ are smooth vector fields, state space is $Q = (x, y, q_0, q_1, q_2, q_3) \in M = \mathbb{R}^2 \times S^3$, and control $u = (u_1, u_2) \in \mathbb{R}^2$ is unbounded.

Since considered system is left-invariant problem on Lie Group $\mathbb{R}^2 \times S^3$ the motion planning problem for any boundary conditions is reduced to fixed initial position and arbitrary final one:

$$Q(0) = Q_0 = (0, 0, 1, 0, 0, 0), \qquad Q(t_1) = Q_1.$$

To apply the motion planning algorithm we choose local chart

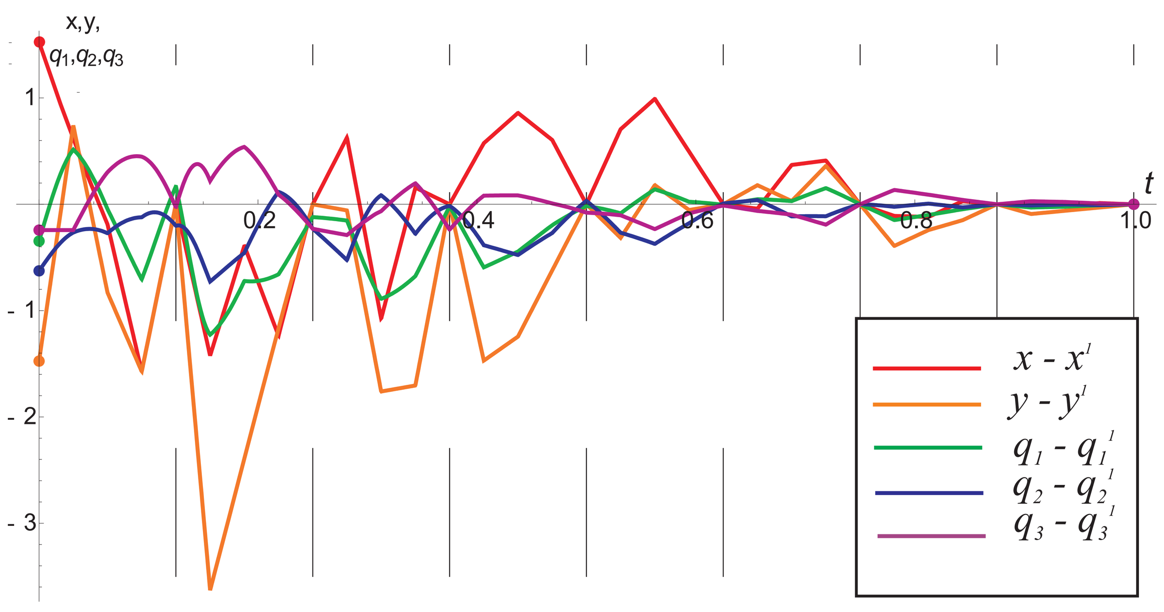
$$q_0 = \sqrt{1 - q_1^2 - q_2^2 - q_3^2} > 0.$$

MotionPlanning Package: Examples

Rolling the sphere using piecewise constant control from initial configuration $Q_0 = (0, 0, 0, 0, 0)$ to desired configuration

$$Q_1 = (-1.525, 1.475, 0.346, 0.626, 0.242)$$

with precision $\varepsilon = 10^{-5}$



Iterations: 8

Transferring the car with two trailers from initial configuration

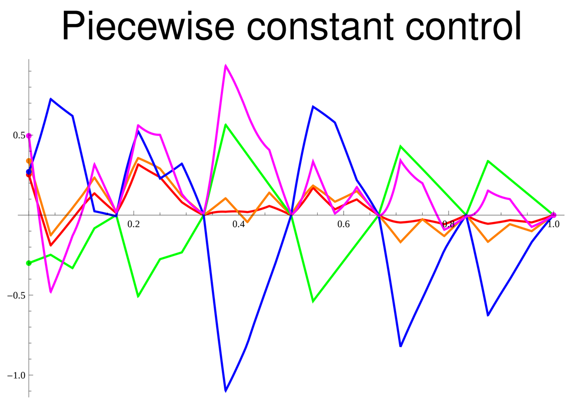
$$Q_0 = (0, 0, \frac{\pi}{4}, \frac{\pi}{4}, -\frac{\pi}{4})$$

to desired configuration

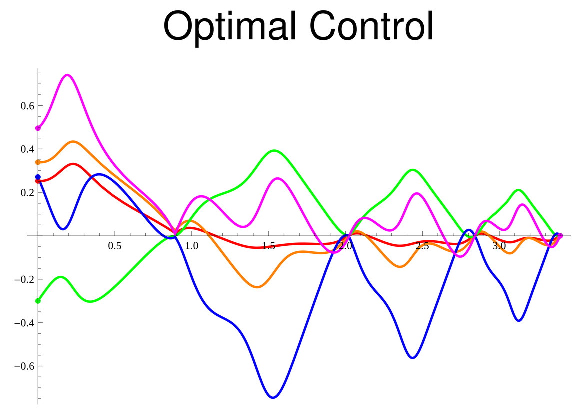
$$Q_1 = (-0.252, -0.339, 1.085, 0.514, -1.281)$$

with precision $\varepsilon = 10^{-3}$

Piecewise constant control | Optimal Control



6 iterations



4 iterations

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Conclusion

We presented the iterative algorithm for solving motion planning problem (1), (2) whith any necessary precision. It has been implemented in a parallel software MotionPlaning.m. The software has been tested on two applications (problem of rolling of a sphere on a plane without slipping and twisting and the problem of steering the mobile robot with two trailers). In cases where the boundary conditions were not too far from each other, the software has been successfully solving the control problem. In cases of distant boundary conditions algorithm does not converge, which corresponds to the theoretical basis of the method (nilpotent approximation is the local approximation of the original system). In the future we plan to expand the functionality for solving the tasks with distant boundary conditions through its reduction to the sequence of local problems. Currently MotionPlaning.m is a convenient and reliable way to solve the local problem (1), (2).

NON-LIPSCHITZ POINTS AND THE SBV REGULARITY OF THE MINIMUM TIME FUNCTION

ITN SADCO
Initial Training Network
Sensitivity Analysis for Deterministic Controller Design

Giovanni Colombo, Nguyen T. Khai, Nguyen V. Luong
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INTRODUCTION

Consider the control dynamics:

$$\begin{cases} \dot{y} = F(y) + G(y)u \\ u \in \mathcal{U} \\ y(0) = x \in \mathbb{R}^N, \end{cases} \quad , \qquad (1)$$

where F, G are smooth enough and the control set $\mathcal{U} \subset \mathbb{R}^M$ is compact.

The **minimum time** $T(x)$ to reach the origin from x:

$$T(x) = \inf\{t : y(t) = 0, y \text{ is a solution of } (1)\}$$

In general, T is **nonsmooth** and even **non-Lipschitz**. Under a controllability assumption, T is **semiconcave/convex** and thus satisfies several regularity properties. In particular,

- T is a.e. twice differentiable.
- The singular set of T has a structure.
- T is locally BV (Bounded Variation).

By weakening the controllability assumptions (e.g., assuming T merely continuous), one can prove that T, although not Lipschitz, satisfies essentially the same properties of a semiconcave/convex function, including **a.e. twice differentiable** and **locally BV**. Under such assumption, we show that non-Lipschitz points of T lie exactly where the Hamiltonian vanishes. Our main result is the $\mathcal{H}^{N-1}$-rectifiability of the set $\mathcal{S}$ of non-Lipschitz points of T for the linear single input case and the $\mathcal{H}^1$ -rectifiability for the nonlinear two dimensional case. As a consequence we obtain that **T is locally SBV** .

CONCEPTS AND NOTIONS

- $\mathcal{R}_\tau = \{x : T(x) \le \tau\}$.
- A closed set $K \subset \mathbb{R}^N$ is said to have **positive reach** iff there exists a continuous function $f : K \to [0, +\infty)$ such that for all $x, y \in K$ and $v \in N_K(x)$
$$\langle v, y - x \rangle \le f(x)\|v\|\,\|y - x\|^2.$$
- Let $0 \le k < \infty$ and let $\mathcal{H}^k$ denote the Hausdorff k-dimensional measure. Let E be $\mathcal{H}^k$-measurable. We say that E is **countably $\mathcal{H}^k$-rectifiable** if there exist countably many sets $A_i \subseteq \mathbb{R}^k$ and countably many Lipschitz functions $f_i : A_i \to \mathbb{R}^N$ be such that
$$\mathcal{H}^k\left(E \setminus \bigcup_{i=1}^{\infty} f_i(A_i)\right) = 0.$$
- A BV function φ is **SBV (Special Bounded Variation)** if its distributional derivative $D\varphi$ has no Cantor part.

MINIMIZED HAMILTONIAN AND NON-LIPSCHITZ POINTS

The Minimized Hamiltonian: $h(x, \zeta) = \langle F(x), \zeta \rangle + \min_{u \in \mathcal{U}} \langle G(x)u, \zeta \rangle$.
We prove: Under assumptions which imply epi(T) has positive reach, T is non-Lipschitz at x iff there exists $0 \ne \zeta \in \mathbb{R}^N$ such that $h(x, \zeta) = 0$ and $\zeta \in N_{\mathcal{R}_{T(x)}}(x)$.

RESULTS FOR LINEAR SYSTEMS IN $\mathbb{R}^N$

Consider the linear control dynamics: $\dot{x} = Ax + bu, \quad |u| \le 1,$ where $A \in \mathbb{M}_{N \times N}$, $b \in \mathbb{R}^N$, satisfies the Kalman rank condition $\text{rank}[b, Ab, \dots, A^{N-1}b] = N$.
Then epi(T) has positive reach and the set of all non-Lipschitz points of T is

$$\mathcal{S} = \left\{ x : \exists r > 0, \zeta \in \mathbb{S}^{N-1} \text{ such that } x = \int_0^r e^{A(t-r)} b \,\text{sign}\left(\langle \zeta, e^{At} b \rangle\right) dt \text{ and } \langle \zeta, b \rangle = 0 \right\}$$

We prove also:

- $\mathcal{S}$ is closed, countably $\mathcal{H}^{N-1}$-rectifiable.
- $T \in SBV_{\text{loc}}(\mathbb{R}^N)$.
- (Propagation result) For $\mathcal{H}^{N-1}-$ a.e $x \in \mathcal{S}$ there exists a neighborhood V of x such that
$$\mathcal{H}^{N-1}(V \cap \mathcal{S}) > 0.$$

RESULTS FOR NONLINEAR SYSTEMS IN $\mathbb{R}^2$

Consider the nonlinear control dynamics: $\dot{x} = F(x) + G(x)u, \quad |u| \le 1,$ where $F, G : \mathbb{R}^2 \to \mathbb{R}^2$ are of class $\mathcal{C}^{1,1}$ satisfying $F(0) = 0$, $G(0) = 0$ and $\text{rank}[G(0), DF(0)G(0)] = 0$.

There exists $\mathcal{T} > 0$ depending only on the data of the dynamics such that for all $0 < \tau < \mathcal{T}$, epi(T) has positive reach in $\mathcal{R}_\tau$ and the set of all non-Lipschitz points of T within $\mathcal{R}_\tau$ is

$$\mathcal{S} = \{x \in \mathcal{R}_\tau : \exists \zeta \in \mathbb{S}^1 \cap N_{\mathcal{R}_{T(x)}}(x) \text{ such that } h(x, \zeta) = 0\}$$

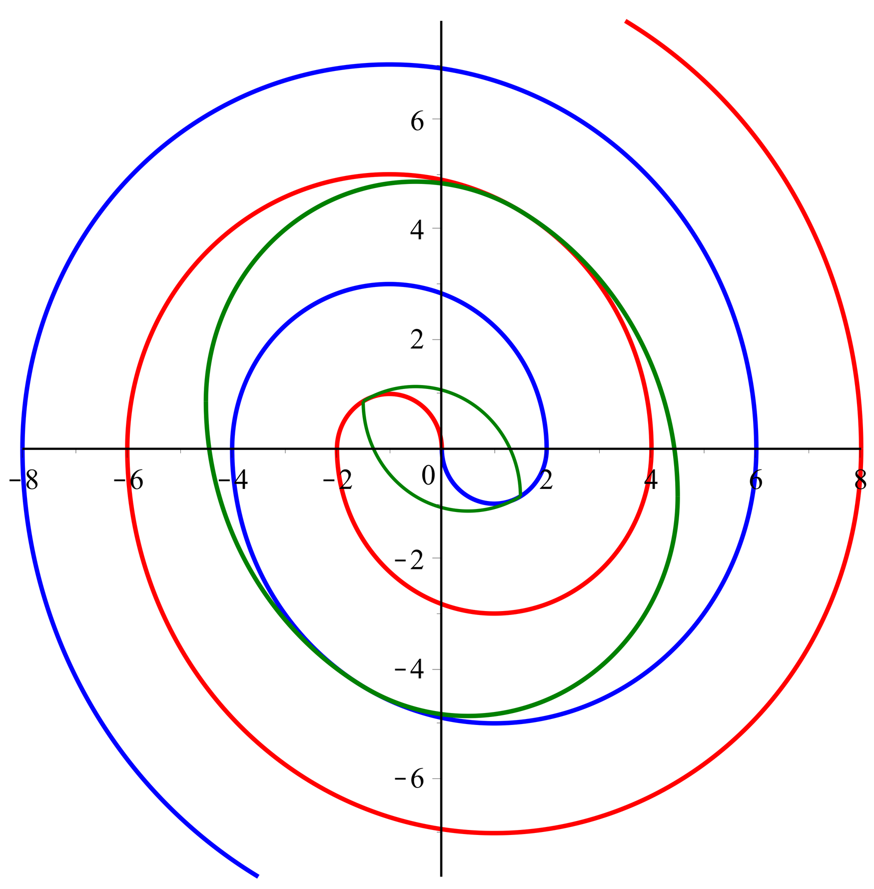
We prove also:

- $\mathcal{S}$ is closed, countably $\mathcal{H}^1$ - rectifiable.
- $T \in SBV_{\text{loc}}(\text{int}(\mathcal{R}_\mathcal{T}))$.
- (Propagation result) For all $\bar{x} \in S$ there exists $\delta > 0$ such that $\mathcal{H}^1\left(S \cap B(\bar{x}, \delta)\right) > 0$.

EXAMPLES

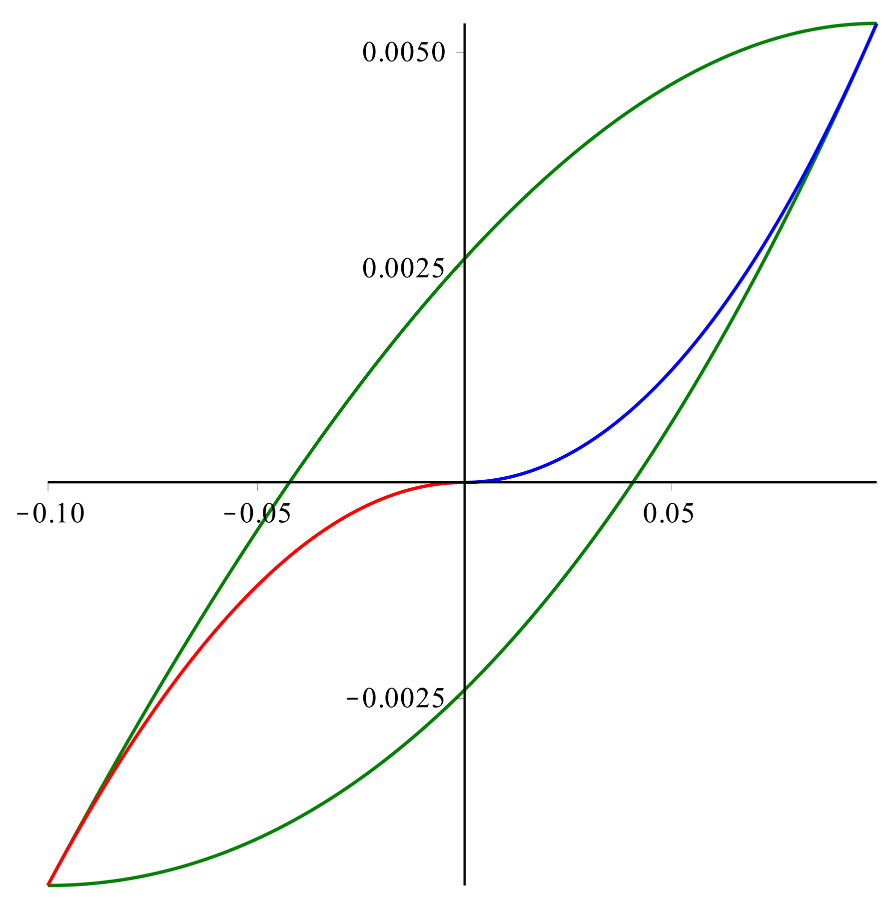
Consider the linear control system:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x + u \end{cases}, u \in [-1, 1]$$



Consider the nonlinear control system:

$$\begin{cases} \dot{x} = x^2 - u \\ \dot{y} = -x^2 - x \end{cases}, u \in [-1, 1]$$



Remark:

- The set $\mathcal{S}$ of non-Lipschitz points of T consists of two optimal trajectories (the **red** and **blue** curves) starting from the origin.
- The non-Lipschitz trajectories are tangent to sublevels of T (the **green** curves).

FUTURE WORK

Extending the above results to higher dimensional nonlinear control systems.

Sensitivity analysis for relaxed optimal control problems with final-state constraints

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Abstract

We consider a family of optimal control problems with final-state constraints parameterized by a nonnegative variable $\theta \geq 0$. The value function is denoted by $V(\theta)$. We consider bounded strong solutions to these problems, *ie*, optimal solutions in a small neighborhood in L^∞ for trajectories and a large bounded neighborhood for the controls. Our aim is to obtain a second-order expansion of $V(\theta)$ near 0. By introducing relaxed controls, we are able to deal with a wide class of perturbations and we obtain sharp estimates.

1 Formulation of the problem

1.1 Setting

For a control u in $L^\infty([0,T],\mathbb{R}^m)$ and $\theta \geq 0$, consider the trajectory $y[u,\theta]$ solution of the following differential system:

$$\begin{cases} \dot{y}_t = f(u_t, y_t, \theta), & \text{for a. a. } t \text{ in } [0,T], \\ y_0 = y^0(\theta). \end{cases}$$

We set $K = \{0_{n_E}\} \times \mathbb{R}_+^{n_I}$. The family of optimal control problems that we consider is the following:

$$\underset{u \in L^\infty([0,T],\mathbb{R}^n)}{\text{Min}} \phi(y_T[u,\theta]), \quad \text{s.t. } \Phi(y_T[u],\theta) \in K.$$

A control $\overline{u}$ is said to be a **bounded strong solution** for the reference problem (with $\theta = 0$) if for all $R > ||u||_\infty$, there exists $\eta > 0$ such that $\overline{u}$ is solution to the localized problem

$$\underset{u \in L([0,T],B_R)}{\text{Min}} \phi(y_T[u,0]), \quad \text{s.t. } \Phi(y_T[u],0) \in K, \ ||y[u,0] - \overline{y}||_\infty \leq \eta, \qquad (\mathcal{P})$$

where B_R is the ball of radius R and $\overline{y} = y[\overline{u},0]$. Now, we fix $\overline{u}$, R, and η.

1.2 Relaxation

Let X be a closed subset of $\mathbb{R}^m$, we denote by $\mathcal{P}(X)$ the set of probabilities on X. The space of **Young measures** $\mathcal{M}^Y(X)$ is the set of measurable mapping from $[0,T]$ to $\mathcal{P}(X)$ [5]. We equip this space with:

- ▷ the weak-$*$ topology,
- ▷ the narrow topology,
- ▷ the usual L^p-distance of transportation theory, denoted by d_p.

For example, a sequence of controls oscillating increasingly fast between to values a and b converges weakly-$*$ to $\mu_t = (\delta_a + \delta_b)/2$. We denote by $\overline{\mu}$ the Young measure such that $\overline{\mu}_t = \delta_{\overline{u}_t}$. For μ in $\mathcal{M}^Y(B_R)$, we denote by $y[\mu,\theta]$ the solution to

$$\begin{cases} \dot{y}_t = \int_{B_R} f(u, y_t, \theta)\, \mathrm{d}\mu_t(u), & \text{for a. a. } t \text{ in } [0,T], \\ y_0 = y^0(\theta). \end{cases}$$

We consider the family of relaxed problems with value function $V(\theta) =$

$$\underset{\mu \in \mathcal{M}^Y(B_R)}{\text{Min}} \phi(y_T[\mu,\theta]), \quad \text{s.t. } \Phi(y_T[\mu],\theta) \in K, \ ||y[\mu,\theta] - \overline{y}||_\infty \leq \eta. \qquad (\mathcal{P}^Y)$$

1.3 Pontryagin linearization

For a given μ in $M^Y(B_R)$, we define the **Pontryagin linearization** $\xi[\mu]$ as follows:

$$\begin{cases} \dot{\xi}_t[\mu] = f_y(\overline{u}_t, \overline{y}_t)\xi_t[\mu] + \int_{U_R} f(u, \overline{y}_t) - f(\overline{u}_t, \overline{y}_t)\, \mathrm{d}\mu_t(u), \\ \xi_0[\mu] = 0. \end{cases}$$

We denote by ξ^θ the solution to

$$\begin{cases} \dot{\xi}_t^\theta = f_y[t]\xi_t^\theta + f_\theta[t], & \text{for a. a. } t \text{ in } [0,T], \\ \xi^\theta = y_\theta^0(0). \end{cases}$$

The following estimate holds

$$||y[\mu,\theta] - (\overline{y} + \xi[\mu] + \theta\xi^\theta)||_\infty = O(d_1(\mu,\overline{\mu})^2 + \theta^2).$$

1.4 Qualification

We set $\mathcal{R}_T = \{\xi_T[\mu], \mu \in \mathcal{M}^Y(B_R)\}$ and we denote by $\mathcal{C}(\mathcal{R}_T)$ the smallest closed cone containing $\mathcal{R}_T$. We assume that the following **qualification condition** holds: there exists $\varepsilon > 0$ such that

$$B_\varepsilon \subset \Phi(\overline{y}_T, 0) + \Phi_{y_T}(\overline{y}_T, 0)\mathcal{C}(\mathcal{R}_T) - K.$$

Theorem (Metric regularity)**.** *There exist $\tilde{\theta} > 0$, $\delta > 0$ and $C > 0$ such that for all θ in $[0,\tilde{\theta}]$ and for all μ in $\mathcal{M}^Y(B_R)$ satisfying $d_1(\mu,\overline{\mu}) \leq \delta$, there exists a control μ' such that*

$$\Phi(y_T[\mu',\theta],\theta) \in K \quad \textit{and} \quad d_1(\mu,\mu') \leq C \cdot \textit{dist}(\Phi(y_T[\mu,\theta],\theta), K).$$

1.5 Motivations for the relaxation

It can be checked that

- ▷ $\mathcal{M}^Y(B_R)$ is weakly-$*$ compact
- ▷ $L([0,T],B_R)$ is weakly-$*$ dense in $\mathcal{M}^Y(B_R)$
- ▷ $y[\mu,\theta]$ is weakly-$*$ continuous.

Therefore,

- ▷ the relaxed problems posseses optimal solutions
- ▷ if $\overline{\mu}$ is the only control μ such that $y[\mu] = \overline{y}$, then problems $(\mathcal{P})$ and $(\mathcal{P}^Y)$ have the same value.

2 Methodology of sensitivity analysis

Following [3], we describe the methodology used in an abstract framework:

$$V(\theta) = \underset{x \in H}{\text{Min}} \ f(x,\theta) \quad \text{s.t. } g(x,\theta) \in K, \qquad (P_\theta)$$

where H is a Hilbert space and K stands for inequalities and equalities. The Lagrangian is

$$L(x,\lambda,\theta) = f(x,\theta) + \langle \lambda, g(x,\theta) \rangle.$$

Let $\overline{x}$ be an optimal solution to (P_0) and Λ be the set of Lagrange multipliers associated.

2.1 First-order upper estimate

Let d in H be such that $g'(\overline{x},0)(d,1) \in T_K(g(\overline{x},0))$. With a regularity theorem, we construct a feasible sequence $x_\theta = \overline{x} + \theta d + o(\theta)$. Therefore, the linear problem

$$\underset{d \in H}{\text{Min}} f'(\overline{x},0)(d,1) \quad \text{s.t. } g'(\overline{x},0)(d,1) \in T_K(g(\overline{x},0)), \qquad (LP)$$

provides the upper estimate $V(x) \leq V(0) + \theta\,\mathrm{Val}(LP) + o(\theta)$. Moreover, the dual of (LP) is

$$\underset{\lambda \in \Lambda}{\text{Max}} L_\theta(\overline{x},\lambda,0). \qquad (LD)$$

2.2 Second-order upper estimate

Let d be a solution to (LP). We define

$$\begin{cases} \underset{h \in H}{\text{Min}} & f_x(\overline{x},0)h + \frac{1}{2}f''(\overline{x},0)(d,1)^2 \\ \text{s.t.} & g_x(\overline{x},0)h + \frac{1}{2}g''(\overline{x},0)(d,1)^2 \in T_K^2(g(\overline{x},0), g'(\overline{x},0)d). \end{cases} \qquad (QP)$$

The dual of this problem is

$$\underset{\lambda \in S(LD)}{\text{Max}} L_{(x,\theta)^2}(\overline{x},\lambda,0)(d,1)^2. \qquad (QD)$$

Finally, we obtain the upper expansion of $V(\theta)$

$$V(0) + \theta\Big(\mathrm{Val}(LP)\Big) + \theta^2\Big(\underset{d \in S(LP_\theta)}{\text{Min}} \ \underset{\lambda \in S(LD)}{\text{Max}} L_{(x,\theta)^2}(\overline{x},\lambda,0)(d,1)^2\Big) + o(\theta^2).$$

2.3 Rate of convergence of solutions

We consider a **strong sufficient second-order condition:** there exists $\alpha > 0$ such that for all h in the critical cone,

$$\sup_{\lambda \in S(LD_\theta)} L_{xx}(\overline{x},\lambda,0)h^2 \geq \alpha|h|^2.$$

If this condition is satisfied, then the solutions x^θ to (P_θ) are such that

$$|x^\theta - \overline{x}| = O(\theta).$$

Moreover, the sequence $(x^\theta - \overline{x})/\theta$ has all its limit points in $S(LP)$.

2.4 Second-order lower estimate

A second order expansion follows from a Taylor expansion: for all λ in $S(LD)$,

$$\begin{aligned} V(\theta) - V(0) &= f(x^\theta) - f(\overline{x}) \\ &\geq L(x^\theta,\lambda,\theta) - L(\overline{x},\lambda,0) \\ &= \theta L_\theta(\overline{x},\lambda,0) + \frac{\theta^2}{2}\Big(L_{(x,\theta)^2}(\overline{x},\lambda,0)\big(\frac{x^\theta - \overline{x}}{\theta},1\big)^2\Big) + o(\theta^2) \\ &\geq \theta L_\theta(\overline{x},\lambda,0) + \frac{\theta^2}{2}\Big(\underset{d \in S(LP)}{\text{Min}} L_{(x,\theta)^2}(\overline{x},\lambda,0)(d,1)^2\Big) + o(\theta^2). \end{aligned}$$

3 Upper estimates

3.1 First-order upper estimate

For the optimal control problems, we consider perturbations of this form:

$$\mu^\theta = (1-\theta)\overline{\mu} + \theta\mu,$$

where the addition is **the addition of measures**. We have

$$y[\mu^\theta,\theta] = \overline{y} + \theta(\xi[\mu] + \xi^\theta) + o(\theta).$$

The equivalent of problem (LP) is now:

$$\underset{\xi \in \mathcal{C}(\mathcal{R}_T)}{\text{Min}} \phi'(\overline{y}_T,0)(\xi + \xi_T^\theta, 1) \quad \text{s.t. } \Phi'(\overline{y}_T,0)(\xi + \xi_T^\theta, 1) \in T_K(\Phi(\overline{y}_T,0)).$$

Let us define:

- ▷ the end-point Lagrangian, $\Phi[\lambda](y,\lambda,\theta) = \phi(y,\theta) + \lambda\Phi(y,\theta)$,
- ▷ the Hamiltonian, $H[p](u,y,\theta) = \langle p, f(u,y,\theta) \rangle$,
- ▷ the costate p^λ associated with λ in $N_K(\Phi(\overline{y}_T,0))$, the solution to

$$\begin{cases} \dot{p}_t = & -H_y[p_t](\overline{u}_t,\overline{y}_t) \\ p_T = & \Phi_{y_T}[\lambda](\overline{y}_T,\lambda,0). \end{cases}$$

- ▷ **Pontryagin multipliers** Λ^P, the set of λ in $N_K(\Phi(\overline{y}_T,0))$ such that for almost all t, $u \mapsto H[p_t^\lambda](u,\overline{y}_t,0)$ is minimized by $\overline{u}_t$.

The dual of problem (LP) is

$$\underset{\lambda \in \Lambda}{\text{Max}} \Big\{p_0^\lambda y_\theta^0(0) + \int_0^T H_\theta[p_t^\lambda](\overline{u}_t,\overline{y}_t,0)\,\mathrm{d}t + \Phi_\theta(\overline{y}_T,0)\Big\}. \qquad (LD)$$

3.2 Second-order upper estimate

Unfortunately, problem (LP) does not have necessarily solutions. We consider the linearization associated with the perturbation $\overline{u} + \theta v$. We denote by $z[v]$ the solution of

$$\begin{cases} \dot{z}_t[v] & = f_{u,y}(\overline{u}_t,\overline{y}_t,0)(v_t, z_t[v]), \\ z_0[v] & = y_0^\theta(0), \end{cases}$$

and we set $z^1[v] = z[v] + \xi^\theta$. This definition extends to ν in $\mathcal{M}_2^Y$, the space of Young measures with a finite L^2-norm. The standard linearized problem is

$$\underset{\nu \in \mathcal{M}_2^Y}{\text{Min}} \phi'(\overline{y}_T,0)(z_T^1[v],1) \text{ s.t. } \Phi'(\overline{y}_T,0)(z_T^1[v],1) \in T_K(\Phi(\overline{y}_T,0)). \qquad (SLP)$$

Now, $\lambda \in N_K(\Phi(\overline{y}_T,0))$ is said to be a **Lagrange multiplier** if for almost all t,

$$H_u[p_t^\lambda](\overline{u}_t,\overline{y}_t) = 0.$$

The set of Lagrange multipliers is denoted by Λ^L. Note that $\Lambda^P \subset \Lambda^L$. The dual of (SLP) is:

$$\underset{\lambda \in \Lambda^L}{\text{Max}} L_\theta(\overline{u},\overline{y},\lambda,0). \qquad (SLD)$$

Now we assume that:

$$\mathrm{Val}(SLP) = \mathrm{Val}(LP).$$

Consider a solution ν to (SLP). Considering a perbutation of the form

$$\mu^\theta = (1-\theta^2)(\overline{u} + \theta v) + \theta^2\mu,$$

we obtain a second-order problem whose dual is the following:

$$\underset{\lambda \in S(LD)}{\text{Max}} \Omega^\theta[\lambda](v), \qquad (QD(\nu))$$

where Ω^θ is defined by

$$\begin{aligned} \Omega^\theta[\lambda](\nu) = {} & p_0^\lambda y_{\theta\theta}^0(0) + \Phi''[\lambda](\overline{y}_T,0)(z_T^1[\nu],1) \\ & + \int_0^T \int_{\mathbb{R}^m} H''[p_t^\lambda](\overline{u}_t,\overline{y}_t,0)(u, z_t^1[\nu],1)^2\,\mathrm{d}\mu_t(u)\,\mathrm{d}t. \end{aligned}$$

Theorem. *The following estimate holds:*

$$V(\theta) \leq V(0) + \theta\,\mathit{Val}(LP_\theta) + \frac{\theta^2}{2}\Big(\underset{\nu \in S(SLP)}{\mathit{Min}} \ \underset{\lambda \in S(LD)}{\mathit{Max}} \Omega[\lambda](\nu)\Big) + o(\theta^2).$$

4 Lower estimate

The critical cone C is the set of ν in $M_Y^2(\mathbb{R}^m)$ such that

$$\begin{aligned} &\phi_{y_T}(\overline{y}_T,0)z_T[\nu] \leq 0, \\ &\Phi_{y_T}(\overline{y}_T,0)z_T[\nu] \in T_K(\Phi(\overline{y}_T,0)). \end{aligned}$$

We also define the quadratic form $\Omega[\lambda](\nu)$ by

$$\begin{aligned} \Omega[\lambda](\nu) = {} & \Phi_{(y_T)^2}[\lambda](\overline{y}_T,0)(z_T[\nu]) \\ & + \int_0^T \int_{\mathbb{R}^m} H_{(u,y)^2}[p_t^\lambda](\overline{u}_t,\overline{y}_t,0)(u, z_t[\nu])^2\,\mathrm{d}\mu_t(u)\,\mathrm{d}t. \end{aligned}$$

The **strong second-order sufficient condition** is: $\exists\alpha > 0$ such that

1. For all μ in $\mathcal{M}^Y(B_R)$,

$$\sup_{\lambda \in S(LD_\theta)} \int_0^T H[p_t^\lambda](u,\overline{y}_t,0) - H[p_t^\lambda](\overline{u}_t,\overline{y}_t,0)\,\mathrm{d}\mu_t(u)\,\mathrm{d}t \geq \alpha d_2(\overline{\mu},\mu).$$

2. For all ν in C, $\sup_{\lambda \in S(LD_\theta)} \Omega[\lambda](\nu) \geq \alpha||\nu||_2^2$.

We consider solutions μ^θ to the perturbed problem.

Theorem. *For all sequence $\theta_k \downarrow 0$, the sequence $\frac{\mu^{\theta_k} \ominus \overline{u}}{\theta_k}$ has a limit point for the narrow topology in $S(SLP)$. Moreover,*

$$V(\theta) \geq V(0) + \theta\,\mathit{Val}(LP_\theta) + \frac{\theta^2}{2}\Big(\underset{\nu \in S(SLP)}{\mathit{Min}} \ \underset{\lambda \in S(LD_\theta)}{\mathit{Max}} \Omega[\lambda](\nu)\Big) + o(\theta^2).$$

Sketch of the proof. Following [1], we decompose a solution μ^k into two controls:

- ▷ $\mu^{A,k}$, accounting for the small variations in $L^\infty-$norm of the control,
- ▷ $\mu^{B,k}$, accounting for the large variations in $L^\infty-$norm of the control, but on a small subset of $[0,T] \times B_R$.

For all λ in $S(LD)$, we have

$$\phi(y_T[\mu^\theta,\theta],\theta) - \phi(\overline{y}_T,0) \geq \Phi[\lambda](y_T[\mu^\theta,\theta],\theta) - \Phi[\lambda](\overline{y}_T,0).$$

Then, we expand the r.h.s. and we neglect the part due to $\mu^{B,k}$.

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A semi-Lagrangian scheme for a first order Mean Field Game problem

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1 Introduction

We consider the following first order Mean Field Game problem

$$\begin{aligned} -\partial_t v(x,t) + \tfrac{1}{2}|Dv(x,t)|^2 &= F(x, m(t)), \text{ in } \mathbb{R}^d \times (0,T), \\ \partial_t m(x,t) - \operatorname{div}\big(Dv(x,t)m(x,t)\big) &= 0, \quad \text{in } \mathbb{R}^d \times (0,T), \\ v(x,T) = G(x, m(T)) \quad \text{for } x \in \mathbb{R}^d \ &, \ \ m(0) = m_0 \in L^\infty(\mathbb{R}^d). \end{aligned} \tag{1}$$

The above equations have been introduced by J.M. Lasry and P. L. Lions in [4, 3] in order to model a deterministic differential game with an infinite number of players. The main assumption is that the players are indistinguishable and each one of them has a small influence on the overall system. Existence of a solution, where the first equation is satisfied in the viscosity sense and the second one in the distributional sense, can be proved under rather general assumptions. The uniqueness is also satisfied if the following assumption holds true

$$\left.\begin{aligned} \int_{\mathbb{R}^d} [F(x,m_1) - F(x,m_2)]\, \mathrm{d}[m_1 - m_2](x) > 0 \quad &\text{for all } m_1,\, m_2 \in \mathcal{P}_1,\ m_1 \neq m_2, \\ \int_{\mathbb{R}^d} [G(x,m_1) - G(x,m_2)]\, \mathrm{d}[m_1 - m_2](x) > 0 \quad &\text{for all } m_1,\, m_2 \in \mathcal{P}_1,\ m_1 \neq m_2. \end{aligned}\right\} \tag{M}$$

In this poster we present

- The semi-discrete in time scheme introduced in [1] (joint work with F. Camilli).
- A fully-discrete semi-Lagrangian scheme introduced in [2] (joint work with E. Carlini) which depends on the discretization parameters $\rho > 0$ and $h > 0$ for the state and time, respectively, and a regularization parameter $\varepsilon > 0$.
- An existence result for the fully discrete scheme.
- In the case $d = 1$ a convergence result.
- A numerical simulation.

For precise assumptions over the data, see [1, 2].

2 The semi-discrete scheme [1] (with F. Camilli)

For $h > 0$ and $N \in \mathbb{N}$, with $Nh = T$, and $t_k := kh$ for $k = 0, \dots, N$, we set

$$\mathcal{K}_h := \left\{ \mu = (\mu(t_k))_{k=0}^N : \ \text{ such that } \mu(t_k) \in \mathcal{P}_1 \quad \text{for all } k = 0, \dots, N \right\}.$$

For $\mu \in \mathcal{K}_h$ and $n = [t/h]$, we define recursively the sequence

$$\begin{aligned} v_h[\mu](x,t_k) &= \inf_{\alpha \in \mathbb{R}^d} \left\{ v_h[\mu](x - h\alpha, t_{k+1}) + \tfrac{1}{2}h|\alpha|^2 \right\} + hF(x, \mu(t_k)), \\ v_h[\mu](x,T) &= G(x, \mu(T)). \end{aligned} \tag{2}$$

Given $x \in \mathbb{R}^d$ and $t_{n_1} \leq t_{n_2}$, the *discrete flow* $\Phi_h[\mu](\cdot, t_{n_1}, \cdot)$ is defined recursively as

$$\begin{aligned} \Phi_h[\mu](x, t_{n_1}, t_{n_2+1}) &:= \Phi_h[\mu](x, t_{n_1}, t_{n_2}) - h\alpha_h[\mu](x, t_{n_2}), \\ \Phi_h[\mu](x, t_{n_1}, t_{n_1}) &:= x, \end{aligned}$$

where for every (x, t_k), the discrete control $\alpha_h[\mu](x, t_k)$ solves (2). It can be proved that $\Phi_h[\mu](x, t_{n_1}, \cdot)$ is uniquely defined a.e. in $\mathbb{R}^d$. Now, we define

$$m_h[\mu](t_n) := \Phi_h[\mu](\cdot, 0, t_n)\sharp m_0.$$

The semi-discrete approximation of (1) is defined as

$$\text{Find } m_h \in \mathcal{K}_h \text{ such that } \ m_h(t_n) = \Phi_h[m_h](\cdot, 0, t)\sharp m_0 \ \text{ for all } n = 0, \dots, N. \tag{3}$$

Theorem 1 *Problem* (3) *admits at least one solution* m_h*. Moreover, if* **(M)** *holds then the solution is unique.*

We also have the following convergence result, which, in particular, provides another proof for the existence of a solution of problem (1).

Theorem 2 *Every limit point of* m_h *(there exists at least one) solves* (1)*. In particular, if* **(M)** *holds we have that* $m_h \to m$ *(the unique solution of* (1)*) in* $C([0,T]; \mathcal{P}_1)$ *and in* $L^\infty\left(\mathbb{R}^d \times [0,T]\right)$*-weak-∗.*

The key elements in the proof are optimal control techniques and the fact that m_0 is absolutely continuous w.r.t. the Lebesgue measure.

3 The fully-discrete scheme [2] (with E. Carlini)

For $h, \rho > 0$, let $\mathcal{G}_\rho := \{x_i = i\rho,\ i \in \mathbb{Z}^d\}$ and $\mathcal{G}^{\rho,h} := \{t_n\}_{n=0}^N \times \mathcal{G}_\rho$ be the *time-space grid*. Given the hypercube $Q(x_i) := [x_i \pm \rho e_1] \times \dots \times [x_i \pm \rho e_d]$, set $\beta_i(x) = 1 - \frac{\|x - x_i\|_1}{\rho}$ if $x \in Q(x_i)$ and 0 if not. Given $\mu \in C([0,T], \mathcal{P}_1)$, define

$$v_i^n = S^{\rho,h}[\mu](v^{n+1}, i, n) \quad \text{and} \quad v_i^N = G(x_i, \mu(T)),$$

where $S^{\rho,h}[\mu]$ is defined as

$$S^{\rho,h}[\mu](w, i, n) := \inf_{\alpha \in \mathbb{R}^d} \left[\sum_{j \in \mathbb{Z}^d} \beta_j(x_i - h\alpha) w_j + \tfrac{1}{2}h|\alpha|^2 \right] + hF(x_i, \mu(t_n)).$$

We set

$$v^{\rho,h}[\mu](x,t) := \sum_{i \in \mathbb{Z}^d} \beta_i(x) v_i^{\left[\frac{t}{h}\right]} \quad \text{for all } (x,t) \in \mathbb{R}^d \times [0,T].$$

Let $\rho \in C_c^\infty(\mathbb{R}^d)$ with $\rho \geq 0$ and $\int_{\mathbb{R}^d} \rho(x) \mathrm{d}x = 1$. For $\varepsilon > 0$, we consider the *mollifier* $\rho_\varepsilon(x) := \frac{1}{\varepsilon^d}\rho\left(\frac{x}{\varepsilon}\right)$ and define

$$v_\varepsilon^{\rho,h}[\mu](\cdot, t) := \rho_\varepsilon * v^{\rho,h}[\mu](\cdot, t) \quad \text{for all } t \in [0,T].$$

Consider the set

$$\mathcal{S} := \left\{ (z_i)_{i \in \mathbb{Z}^d}\ ;\ z_i \in \mathbb{R}_+ \text{ and } \sum_{i \in \mathbb{Z}^d} z_i = 1 \right\}.$$

The coordinates of $\mu \in \mathcal{S}^N$ are denoted as μ_i^k, with $i \in \mathbb{Z}^d$ and $k = 0, ..., N$. Each $\mu \in \mathcal{S}^N$ is identified with $\mu \in C([0,T]; \mathcal{P}_1)$ defined as

$$\mu(x,t) := \frac{1}{\rho^d} \left[\frac{t_{k+1} - t}{h} \sum_{i \in \mathbb{Z}^d} \mu_i^k \mathbb{I}_{E_i}(x) + \frac{t - t_k}{h} \sum_{i \in \mathbb{Z}^d} \mu_i^{k+1} \mathbb{I}_{E_i}(x) \right] \quad \text{if } t \in [t_k, t_{k+1}],$$

where $E_i := [x_i \pm \frac{1}{2}\rho e_1] \times \dots \times [x_i \pm \frac{1}{2}\rho e_d]$. Let us define

$$\Phi_\varepsilon^{\rho,h}[\mu](x_i, t_k, t_{k+1}) := x_i - hDv_\varepsilon^{\rho,h}[\mu](x_i, t_k).$$

We define $m_\varepsilon^{\rho,h}[\mu] \in \mathcal{S}^{N+1}$ recursively as

$$\begin{aligned} (m_\varepsilon^{\rho,h}[\mu])_i^{k+1} &:= \textstyle\sum_j \beta_i \left(\Phi_\varepsilon^{\rho,h}[\mu](x_j, t_k, t_{k+1}) \right) (m_\varepsilon^{\rho,h}[\mu])_j^k, \quad \text{for } i \in \mathbb{Z}^d, \\ (m_\varepsilon^{\rho,h}[\mu])_i^0 &:= \textstyle\int_{E_i} m_0(x) \mathrm{d}x, \quad \text{for } i \in \mathbb{Z}^d \end{aligned}$$

and $m_\varepsilon^{\rho,h}[\mu](x,t)$ is defined as we did with μ above. The key property for our main result, which we are able to prove only in dimension 1, is

Lemma 3.1 *Suppose that* $d = 1$*. Then, there exists* $C > 0$ *(independent of* $(\rho, h, \varepsilon, \mu)$*) such that for any* $i \in \mathbb{Z}^d$ *and* $k = 0, \dots, N-1$*, we have that*

$$\sum_{j \in \mathbb{Z}^d} \beta_i \left(\Phi_\varepsilon^{\rho,h}(x_j, t_k, t_{k+1}) \right) \leq 1 + Ch.$$

Consequently, $m_\varepsilon^{\rho,h}[\mu](\cdot,\cdot)$ *is bounded in* L^∞ *independently of* $(\rho, h, \varepsilon, \mu)$.

We consider the following fully-discretization of (1):

$$\text{Find } \mu \in \mathcal{S}^{N+1} \text{ such that } \ \mu_i^k = (m_\varepsilon^{\rho,h}[\mu])_i^k \ \text{ for all } i \in \mathbb{Z}^d \text{ and } k = 0, \dots, N.$$

We have the following existence result:

Theorem 3 *The fully-discrete problem admits at least one solution.*

Our main result is that we can prove convergence in dimension 1.

Theorem 4 *Suppose that* $d = 1$ *and consider a sequence of positive numbers* $\rho_n, h_n, \varepsilon_n$ *satisfying that* $\rho_n = o(h_n \varepsilon_n)$*,* $h_n = o(\varepsilon_n)$ *and* $\varepsilon_n \to 0$*. Then every limit point of* $m_{\varepsilon_n}^{\rho_n,h_n}$ *(there exists at least one) is a solution of* (1)*. In particular, if* **(M)** *holds we have that* $m_{\varepsilon_n}^{\rho_n,h_n} \to m$ *(the unique solution of* (1)*) in* $C([0,T]; \mathcal{P}_1)$ *and in* $L^\infty(\mathbb{R} \times [0,T])$*-weak-∗.*

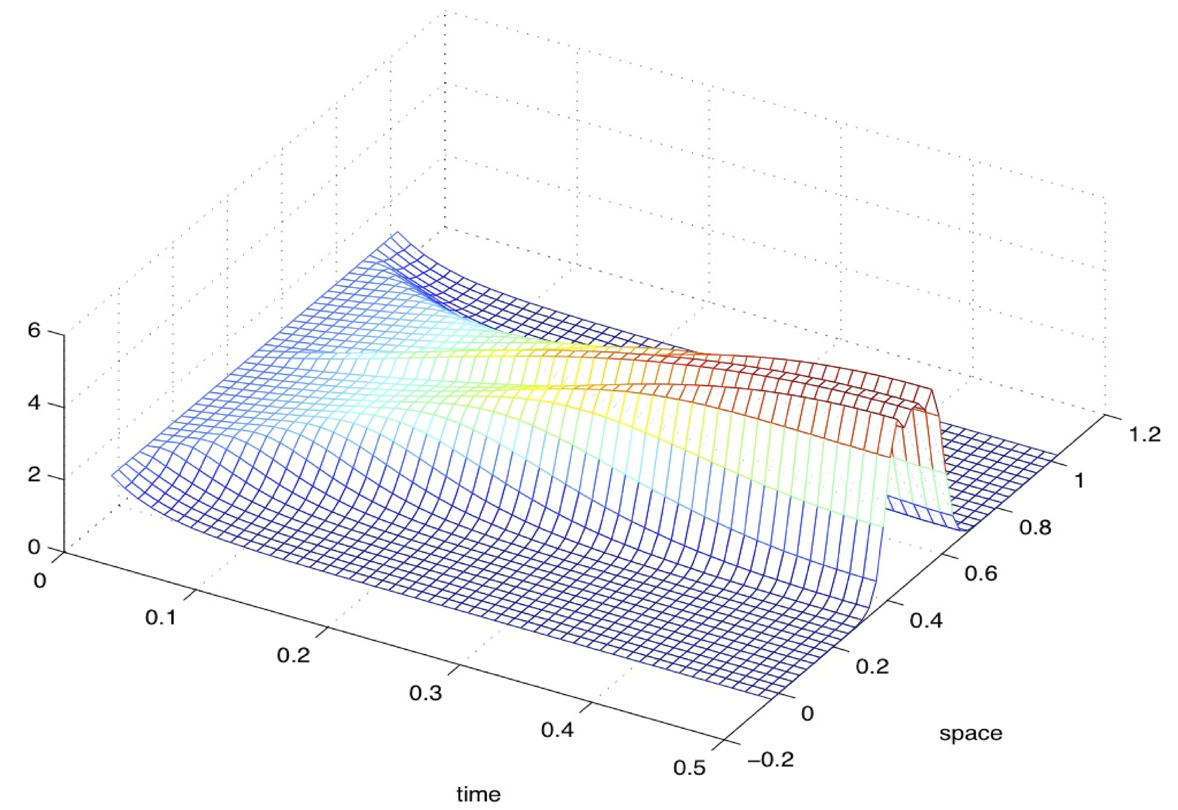
4 A numerical example

Example: [People willing to go to the center but not together]

- Space-time domain: $\Omega \times [0,T] = [0,1] \times [0, 0.05]$
- $F(x,m) = (x - 0.5)^2 + h * (h * m)$, where

$$h(x) = \frac{\hat{h}(x)}{\int_0^1 \hat{h}(y)\mathrm{d}y} \quad \text{and} \quad \hat{h}(x) = e^{-x^2/8}\mathbb{I}_{[-\frac{1}{4},\frac{1}{4}]}.$$

- $G(x,m) = 0$.
- $m_0 \equiv 1$ in $[0,1]$.
- toll$= 10^{-3}$, $\rho = 2.5 \cdot 10^{-2}$ and $h = 0.01$.



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Unexpected Effects of Endogenous Discount Rates on Global Warming Policies

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Problem

Our aim is to study, in the framework of the standard externality problem 'global warming', the consequences of endogenous discounting on the qualitative behavior and compare this approach with alternatives such as exogenous discounting. The Stern report and the ensuing public and academic debate has made clear how crucial discounting is for evaluating anti-global warming measures. A frequently proposed solution, concerning the issue of not weighting the future consequences of today's decisions enough, is hyperbolic discounting, which in turn leads to the problem of time inconsistency. Our basic assumption is that **discounting decreases as the damage increases**, becoming thereby endogenous. This means that decision makers become **more patient when facing the environmental damages** following their high consumption levels.

The Model

An infinitely-lived decision maker benefits from consumption c and suffers damages D from pollution T. Pollution is a stock externality that accumulates with current 'emissions' that are linked to consumption (say of fossil fuels), for simplicity, linearly. Consumption is normalized in units of emissions and the stock of pollution depreciates at a constant rate (δ).

$$\max_{\{c(t)\geq 0\}} \int_0^\infty \left[e^{-\Theta(t)} (u(c(t)) - D(T(t))) \right] \mathrm{d}t,$$

subject to

$$\dot{T}(t) = c(t) - \delta T(t), \forall t, \qquad T(0) = T_0 = 0,$$

and

$$\dot{\Theta}(t) = f(T(t)),$$

where Θ replaces the usual exogenous discount term rt. Thus, Θ is endogenously determined by the discount function $f(T)$, which fulfills that

$$f(0) = r > 0, f > 0 \text{ and } \boldsymbol{f'(T) < 0}, \forall T,$$
$$\text{and } \lim_{T\to\infty} f(T) = 0.$$

$f' < 0$ reflects our assumption, that the discount rate decreases with respect to the pollution stock.

Further Assumptions: The felicity function $u : \mathbb{R}_+ \to \mathbb{R}$ is at least twice differentiable, $u'(c) > 0, u''(c) < 0, \forall c$. Inada conditions: $\lim_{c\to 0} u'(c) = \infty$, and $\lim_{c\to\infty} u'(c) = 0$. The damage function $D : \mathbb{R}_+ \to \mathbb{R}_+$ is at least twice differentiable, $D'(T) > 0, D''(T) > 0$. Inada-type conditions: $D'(0) = 0$, and $\lim_{T\to\infty} D'(T) = \infty$.

Optimality Conditions

The present value Hamiltonian is defined as follows

$$\mathcal{H} = (u(c) - D(T))e^{-\Theta} + \lambda [c - \delta T] - \mu f(T),$$

leading to the necessary optimality conditions

$$\mathcal{H}_c = \lambda + u'e^{-\Theta} = 0,$$
$$\dot{\lambda} = D'(T)e^{-\Theta} + \lambda\delta + \mu f'(T),$$
$$\dot{\mu} = -(u(c) - D(T))\, e^{-\Theta},$$
$$\lim_{t\to\infty} \mathcal{H}(t) = 0.$$

Solving the Model

The dynamics of $\dot{c}$ are found by differentiating $\mathcal{H}_c$ with respect to time and replacing the costate $\lambda = -u'(c)e^{-\Theta}$. Trough making use of the autonomy of $\mathcal{H}$, or equivalently $\dot{\mathcal{H}} = \mathcal{H}_t = 0$, we can eliminate the costate μ. We arrive at the following system of two equations,

$$\dot{c} = \frac{u'(c)}{u''(c)} \left(f(T) + \delta - \frac{D'(T)}{u'(c)} \right) - \frac{1}{u''(c)} \frac{f'(T)}{f(T)} \left(u(c) - D(T) - \dot{T}u'(c) \right), \qquad (1)$$
$$\dot{T} = c - \delta T.$$

In the counterpart of exogenous discounting, $f(T)$ is replaced by the constant discount rate r, and the colored term does not exist. This raises the following questions: existence of steady states, their properties compared to conventional discounting and the dynamic properties.

Steady States

Existence: (proof not shown here)
The assumptions about $u(c)$ and $D(T)$ guarantee the existence of at least one steady state.
Properties:
Due to the colored term in (1), changing the intercept of the utility function $u(c)$, or the intercept of the damage function $D(T)$ – completely irrelevant in the exogenous setup – can render counter-intuitive qualitative behavior. Using a discount function $f(T) < r, \forall T$, we would expect a smaller endogenous damage level $\bar{T}^n$, and consequently a smaller endogenous consumption level $\bar{c}^n = \delta\bar{T}^n$ in the steady state, since less discounting is equivalent to more patience. In contrast, as shown in Figure 1, we can – but do not have to – end up with **a steady state $\bar{T}^n$ higher than the exogenous steady state $\bar{T}^x$.**

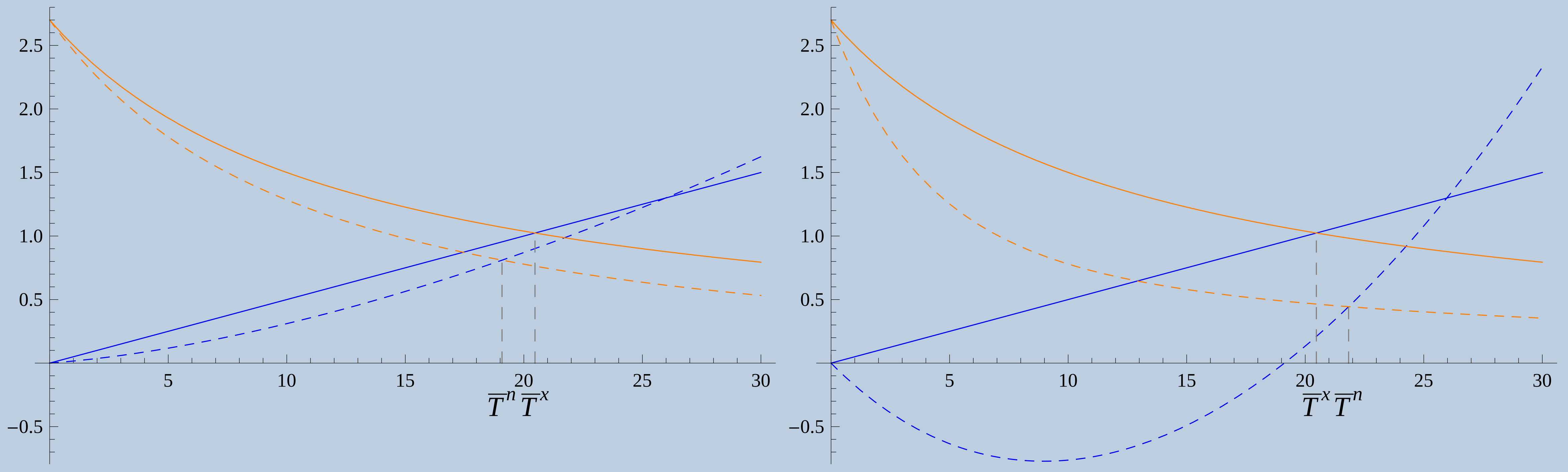

Figure 1: Making use of $\dot{T} = 0$, we replace c by δT in $\dot{c}$. Further, we rearrange $\dot{c} = 0$, so that we arrive at the the endogenous steady state condition $u'(\delta\bar{T}^n)(f(\bar{T}^n) + \delta) = D'(\bar{T}^n) + \frac{f'(\bar{T}^n)}{f(\bar{T}^n)} \left(u(\delta\bar{T}^n) - D(\bar{T}^n) \right)$, or the exogenous steady state condition $u'(\delta\bar{T}^x)(r + \delta) = D'(\bar{T}^x)$ respectively. We print the right and left hand sides of these conditions separately, an intersection characterizes a steady state. The orange graphs illustrate the marginal utility (left hand side of the condition), the blue graphs the marginal damages (right hand side), the solid lines represent the exogenous case, the dashed lines the endogenous one.

In case of exogenous discounting, there exists a unique steady state. Whereas the existence of the colored term in (1) in case of endogenous discounting can trigger **multiple steady states**, as illustrated in Figure 2. Further, we showed that only an uneven number of steady states is possible.

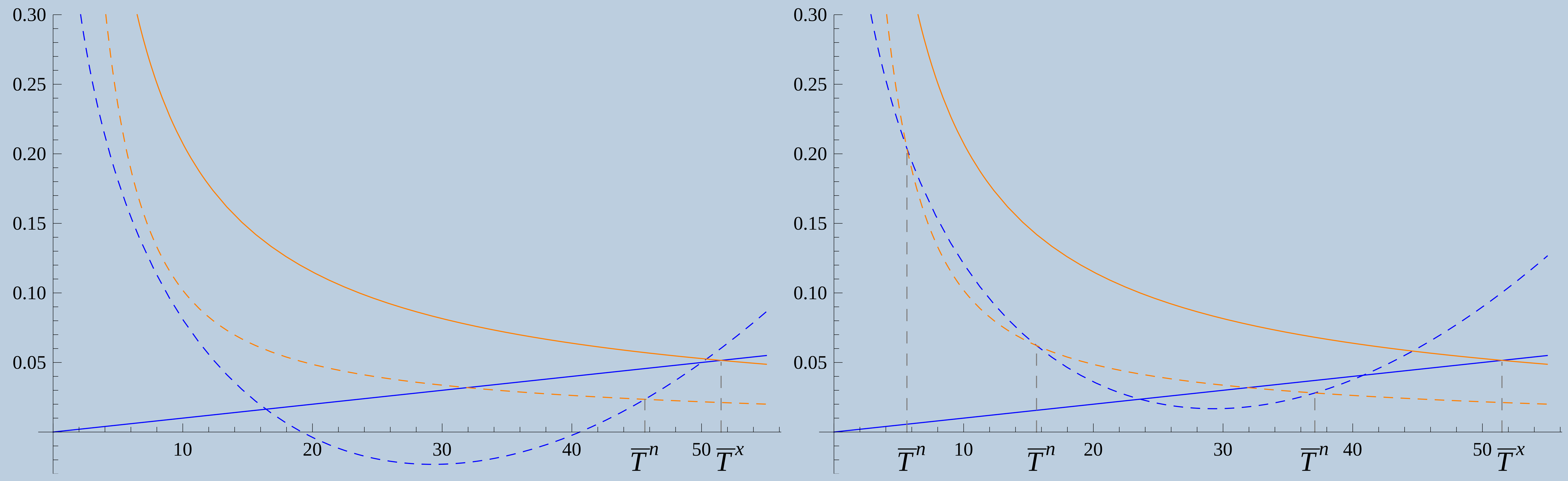

Figure 2: same procedure and same labelling as in Figure 1!

Dynamics

Linearizing the dynamical system around the steady state(s), we use the following Jacobi matrix,

$$J = \begin{bmatrix} \frac{\partial \dot{c}}{\partial c} & \frac{\partial \dot{c}}{\partial T} \\ 1 & -\delta \end{bmatrix} \Rightarrow Det\,(J) = -\left(\delta \frac{\partial \dot{c}}{\partial c} + \frac{\partial \dot{c}}{\partial T} \right) = \frac{\partial \dot{c}}{\partial c} \left(-\frac{\frac{\partial \dot{c}}{\partial T}}{\frac{\partial \dot{c}}{\partial c}} - \delta \right).$$

Thereby, it results that endogenous steady states with an uneven number (and thus also a unique steady state) are always saddlepoint stable. The stability property of (a) steady state(s) with an even number is not that clearly determined, since the sign of the trace is ambiguous. Although it presumably holds that the "middle" steady state(s) are unstable, it can not be arithmetically shown.

Large time behavior of weakly coupled systems of first-order Hamilton-Jacobi equations

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Keywords: weakly coupled systems, large time behavior, Hamilton-Jacobi equations.

1 Problem

We study the large time behavior of systems of Hamilton-Jacobi equations

$$\begin{cases} \dfrac{\partial u_i}{\partial t} + H_i(x, Du_i) + \displaystyle\sum_{j=1}^{m} d_{ij} u_j = 0 \quad (x,t) \in \mathbb{T}^N \times (0, +\infty), \\ u_i(x, 0) = u_{0i}(x), \end{cases} \tag{1}$$

where $d_{ii} \geq 0,\ d_{ij} \leq 0 \text{ for } i \neq j$ and $\sum_{j=1}^{m} d_{ij} = 0$ for all $i = 1, \ldots, m$. We are interested in finding an ergodic constant vector $(c_1, \ldots, c_m) \in \mathbb{R}^m$ and a function $(v_1, \ldots, v_m)$ such that

$$H_i(x, Dv_i) + \sum_{j=1}^{m} d_{ij}(x) v_j = c_i, \quad x \in \mathbb{T}^N, \quad i = 1, \ldots, m \tag{2}$$

and, for all $i = 1, \ldots, m$,

$$u_i(x,t) + c_i t \to v_i(x) \text{ uniformly as } t \text{ tends to infinity,}$$

2 Hypotheses+main result

We assume for $i = 1, \ldots, m$ that

$$\begin{cases} (i) \ \text{The function } p \mapsto H_i(x,p) \text{ is differentiable a.e.,} \\ (ii) \ (H_i)_p \ p - H_i \geq 0 \text{ for a.e. } (x,p) \in \mathbb{T}^N \times \mathbb{R}^N, \\ (iii) \ \text{There exists a, possibly empty, compact set } K \text{ of } \mathbb{T}^N \text{ such that} \\ \quad (a) \ H_i(x,p) \geq 0 \text{ on } K \times \mathbb{R}^N, \\ \quad (b) \ \text{If } H_i(x,p) \geq \eta > 0 \text{ and } d(x,K) \geq \eta, \text{ then } (H_i)_p \ p - H_i \geq \Psi(\eta) > 0. \end{cases}$$

(Result) Assume that $H_i \in C(\mathbb{T}^N \times \mathbb{R}^N)$ satisfies the above hypothesis. Then, the solution $(u_1, \ldots, u_m) \in W^{1,\infty}(^N \times (0,\infty))^m$ of (1) converges uniformly to a solution $(v_1, \ldots, v_m)$ of (2).

3 Applications

A typical example satisfies our result is

$$\begin{cases} \frac{\partial u_1}{\partial t} + |Du_1 + f_1(x)|^2 - |f_1(x)|^2 + u_1 - u_2 = 0, \\ \frac{\partial u_2}{\partial t} + |Du_2 + f_2(x)|^2 - |f_2(x)|^2 + u_2 - u_1 = 0. \end{cases} (x,t) \in^N \times (0, +\infty),$$

where $f_i \in C(\mathbb{T}^N)$. Another example which will be explained through control optimal is given in the next section.

4 Control optimal

Consider the controlled random evolution process (X_t, ν_t) with dynamics

$$\begin{cases} \dot{X}_t = b_{\nu_t}(X_t, a_t), \quad t > 0, \\ (X_0, \nu_0) = (x, i) \in \mathbb{T}^N \times \{1, \ldots, m\}, \end{cases} \tag{3}$$

where the control law $a : [0, \infty) \to A$ is a measurable function (A is a compact subset of some metric space), $b_i \in L^\infty(\mathbb{T}^N \times A; \mathbb{R}^N)$, satisfies

$$|b_i(x,a) - b_i(y,a)| \leq C|x-y|, \qquad x, y \in \mathbb{T}^N,\ a \in A,\ 1 \leq i \leq m. \tag{4}$$

For every a_t and matrix of probability transition $G = (\gamma_{ij})_{i,j}$ satisfying $\sum_{j \neq i} \gamma_{ij} = 1$ for $i \neq j$ and $\gamma_{ii} = -1$, there exists a solution (X_t, ν_t), where $X_t : [0, \infty) \to \mathbb{T}^N$ is piecewise C^1 and $\nu(t)$ is a continuous-time Markov chain with state space $\{1, \ldots, m\}$ and probability transitions given by

$$\mathbb{P}\{\nu_{t+\Delta t} = j \,|\, \nu_t = i\} = \gamma_{ij} \Delta t + o(\Delta t)$$

for $j \neq i$.

We introduce the value functions of the optimal control problems

$$u_i(x,t) = \inf_{a_t \in L^\infty([0,t],A)} \mathbb{E}_{x,i}\{\int_0^t f_{\nu_s}(X_s) ds + u_{0,\nu_t}(X_t)\}, \quad i = 1, \ldots m, \tag{5}$$

where $\mathbb{E}_{x,i}$ denote the expectation of a trajectory starting at x in the mode i, $f_i, u_{0,i} : \mathbb{T}^N \to \mathbb{R}$ are continuous and $f_i \geq 0$.

It is possible to show that the following dynamic programming principle holds:

$$u_i(x,t) = \inf_{a_t \in L^\infty([0,t],A)} \mathbb{E}_{x,i}\{\int_0^t f_{\nu_s}(X_s) ds + u_{\nu_h}(X_h, t-h)\} \qquad 0 < h \leq t.$$

Then the functions u_i satisfy the system

$$\begin{cases} \dfrac{\partial u_i}{\partial t} + \sup\limits_{a \in A} -\langle b_i(x,a), Du_i \rangle + \displaystyle\sum_{j \neq i} \gamma_{ij}(u_i - u_j) = f_i \ (x,t) \in \mathbb{T}^N \times (0, +\infty), \\ u_i(x, 0) = u_{0,i}(x) \qquad\qquad x \in \mathbb{T}^N, \end{cases} i = 1, \cdots m,$$

which has the form (1) by setting $H_i(x,p) = \sup_{a \in A} -\langle b_i(x,a), p \rangle - f_i(x)$ and $d_{ii} = \sum_{j \neq i} \gamma_{ij} = 1$ and $d_{ij} = -\gamma_{ij}$ for $j \neq i$.

We assume that

$$\mathcal{F} = \{x_0 \in^N : f_i(x_0) = 0 \text{ for all } i = 1, \ldots, m\} \neq \emptyset, \tag{6}$$

And the following controllability assumption is satisfied: for every i, there exists $r > 0$ such that for any $x \in^N$, the ball $B(0,r)$ is contained in $\overline{\text{co}}\{b_i(x,A)\}$.

Then our result applies in this case. Roughly speaking, it means that the optimal strategy is to drive the trajectories towards a point x^* of $\mathcal{F}$ and then not to move anymore (except maybe a small time before t). This is suggested by the fact that all the f_i's have minimum 0 at x^* and, at such point, the running cost is 0.

5 References

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