The Maximum Principle and State Constraints

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ITN SADCO
Initial Training Network
Sensitivity Analysis for Deterministic Controller Design

Imperial College London

Outline

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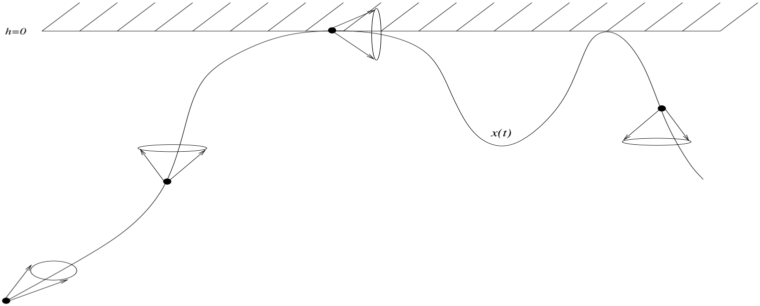
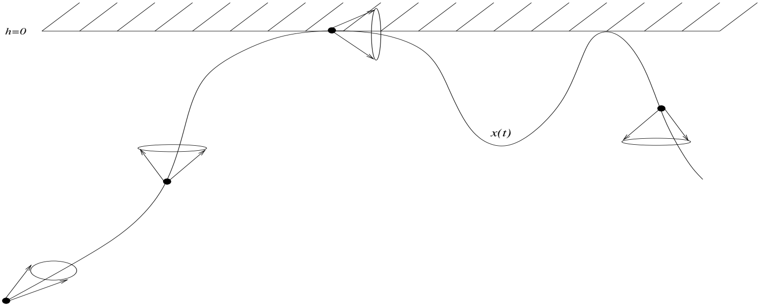
A general constrained optimal control problem



$$\begin{cases} \text{Minimize} \quad g(x(S), x(T)) \\ \dot{x}(t) = f(t, x(t), u(t)) \quad \text{for a.e. } t \in [S, T], \\ (x(S), x(T)) \in E, \\ h(t, x(t)) \leq 0, \text{ for all } t \in [S, T]. \end{cases}$$

Let the admissible pair $(\bar{x}(.), \bar{u}(.))$ be a local minimizer.

A general constrained optimal control problem



$$\begin{cases} \text{Minimize} \quad g(x(S), x(T)) + \int_S^T m(t) h(t, x(t))\, dt \\ \dot{x}(t) = f(t, x(t), u(t)) \quad \text{for a.e. } t \in [S, T], \\ (x(S), x(T)) \in E, \\ h(t, x(t)) \leq 0, \ \text{for all } t \in [S, T]. \end{cases}$$

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Complementary Slackness Condition

$m(t) = 0$ for $t \in \{s : h(s, \bar{x}(s)) < 0\}$

New cost

$$g(x(S), x(T)) + \int_S^T m(t)h(t, x(t))\, dt$$

PMP

- $(q, \lambda) \neq 0$
- $\dot{q}(t) = q(t)f_x(t, \bar{x}(t), \bar{u}(t)) - \lambda h_x(t, \bar{x}(t))m(t)$
- $(q(S), -q(T)) \in N_E(\bar{x}(S), \bar{x}(T)) + \lambda \nabla g(\bar{x}(S), \bar{x}(T))$
- $q(t)f(t, \bar{x}(t), \bar{u}(t)) = \max_{u \in U(t)} q(t)f(t, \bar{x}(t), u)$

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$$\lambda m(t)dt = d\mu(t), \qquad p(t) := q(t) - \int_{[S,t)} h_x(t, \bar{x}(t))\mu(dt)$$

where $supp\{\mu\} \subset \{t : h(t, \bar{x}(t)) = 0\}$ and now
$\dot{p}(t) = q(t)f_x(t, \bar{x}(t), \bar{u}(t))$

R.B. Vinter and G.Pappas. *A Maximum Principle for Nonsmooth Optimal Control Problems with State Constraints.* J. Math. Anal. Appl. pp. 212-232, 1982.

Outline

Vinter-Pappas 1982

A pair (x, u) is an admissible pair when

$$x(0) = c_0, \qquad x(1) \in C_1, \qquad \dot{x}(t) = f(t, x(t), u(t)) \;\; a.e.$$

$$(P) \quad \min \left\{ \int_0^1 L(t, x(t), u(t))\, dt : h(t, x(t)) \leq 0 \right\}$$

Vinter-Pappas 1982

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We approach derivation of optimality conditions through study of the family of problems (P_k), $k \geq 0$

$$(P_k) \quad \min \left\{ \int_0^1 L(t, x(t), u(t))\, dt + k \int_0^1 h^+(t, x(t))\, dt \right\}$$

Hypotheses

(H) $\lim_{k\uparrow\infty} \inf\{P_k\} = \inf\{P\}$.

Define the complete metric space (V, δ):

$$\delta(u, v) := \text{Lebesgue measure}\{t : u(t) \neq v(t)\}$$

$$u \in V \Longrightarrow x_u(.) \text{ is admissible}$$

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and the continuous function

$$J_i(u) := \int_0^1 L(t, x_u(t), u(t))\, dt + k_i \int_0^1 h^+(t, x_u(t))\, dt$$

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By hypothesis (H1) then

$$\epsilon_i^2 := J_i(\bar{u}) - \inf_{u \in V} J_i(u) = \inf\{P\} - \inf\{P_{k_i}\} \downarrow 0$$

Ekeland

$J_i(\bar{u}) \leq \inf_{u \in V} J_i(u) + \epsilon_i^2$

$\exists \{u_i\} \in V$ such that $\delta(u_i, \bar{u}) \leq \epsilon_i$ and

$$u \mapsto J_i(u) + \epsilon_i \delta(u, u_i)$$

attains minimum at u_i.

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(x_i, u_i) is a local minimum for

$$\begin{cases} \min \left\{ \int_0^1 L(t, x(t), u(t)) + k_i \, h^+(t, x(t)) + \epsilon_i \, m_i(t, u(t)) \, dt \right\} \\ \dot{x}(t) = f(t, x(t), u(t)), \quad u(t) \in U(t) \\ x(0) = c_0, \quad x(1) \in C_1, \end{cases}$$

Maximum Principle (state constraint free)

$\exists \{p_i\} \in W^{1,1}$, $\lambda_i \geq 0$ such that

- $|p_i(0)| + \lambda_i \neq 0$
- $-\dot{p}_i(t) = p_i(t) f_x(t, x_i(t), u_i(t)) - \lambda_i L_x(t, x_i(t), u_i(t))$
- $\qquad\qquad -\beta_i(t) h_x(t, x_i(t))$
- $-p_i(1) \in N_{C_1}(x_i(1))$
- $p_i(t) f(t, x_i(t), \bar{u}(t)) - \lambda_i L(t, x_i(t), \bar{u}(t)) = \max_{u \in U(t)} \{p_i(t) f(t, x_i(t), u) - \lambda_i L(t, x_i(t), u)\} \qquad$ a.e. $t \in D_i$

Outline

Summary

- Convergence of measures.
- Conclude the proof of the theorem.
- Discuss on assumption (H) : $\lim_{k\uparrow\infty} \inf\{P_k\} = \inf\{P\}$.
- Difference of formulation (Mayer-Bolza).
- Other methods for handling state constraints.
- Questions.

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OBRIGADO!