Relaxation Approach in Optimal Control Problems

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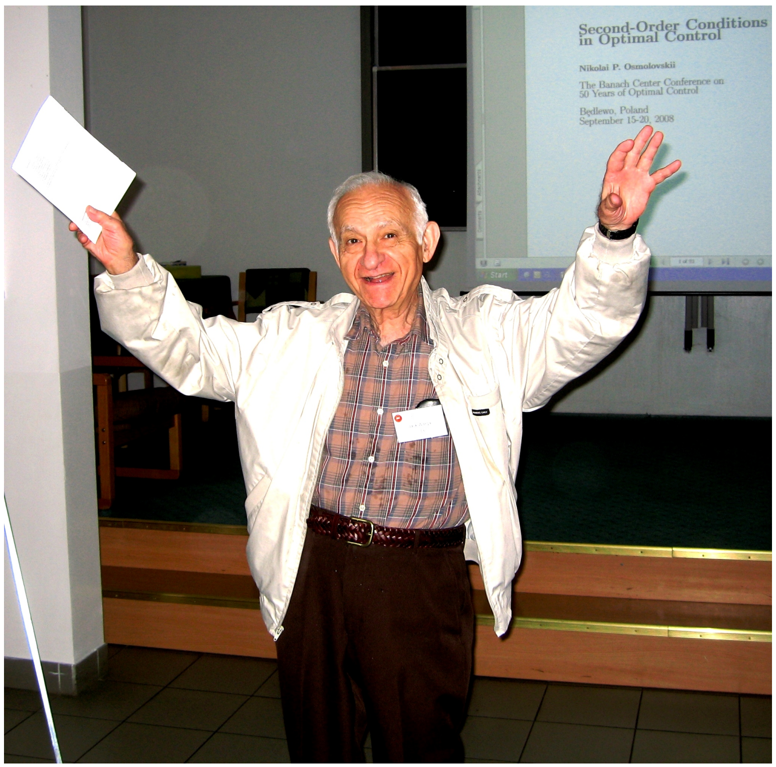
A bit of history...



The term "Relaxation" was conied by J. Warga in the early 60's. In his monography, "Optimal Control of Funcional and Differential Equations", relaxation is investigated and used in several kinds of optimal control problems.

Original Control Problem

Suppose we are considering the following problem:

$$
(P)\ \begin{cases}
\text{minimize}\, g(x(1)) \\
\text{over the absolutely continuos arcs}\, x(\cdot) \\
\text{and the measurable control functions}\, u(\cdot) \text{ s.t.} \\
\dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. } t \in [0,1] \\
u(t) \in U(t) \quad \text{a.e. } t \in [0,1] \\
(x(0), x(1)) \in C_0 \times C_1
\end{cases}
$$

We refer to (P) as original problem, to $u(\cdot)$ as original control function and to the couple $(x(\cdot), u(\cdot))$ as original process.

Existence of a Minimizer

$(H1):$ $f(.,x,u)$ is measurable and $f(t,.,.)$ is continuous. $U(.)$ is a measurable multifunction taking values compact sets. $C_1 \times C_2$ is closed and or C_1 either C_2 is bounded.

$(H2):$ There exist $\varepsilon > 0$, $k(.) \in L^1$ and $c(.) \in L^1$ such that

$$|f(t,x,u) - f(t,x',u)| \leq k(t)|x - x'| \quad \text{and} \quad |f(t,x,u)| \leq c(t)$$

for all $x, x' \in \mathbb{R}^n$, $u \in U(t)$, a.e. $t \in [0,1]$.

Theorem: Suppose that hypotheses $(H1)$ and $(H2)$ are satisfied. Assume furthermore that the set $f(t,x,U(t))$ is convex for all x. Then (P) has a minimizer.

Warga's Relaxation

His approach consists in considering the new integral equation

$$\dot{x}(t) = \int_{U(t)} f(t, x(t), r)\mu(t)(dr) \quad \text{a.e. } t \in [0, 1],$$

where

$$\mu(\cdot) : [0, 1] \to \text{r.p.m.}(U(\cdot)) :=$$

$$= \{\text{(Radon) probability measure on } U(\cdot)\},$$

which replaces the controlled differential equation in (P).

We refer to $\mu(\cdot)$ as relaxed control function and to $(x(\cdot), \mu(\cdot))$ as relaxed process.

Another Approach: Convexification

$$
(P)_{rel} \begin{cases} \text{minimize } g(x(1)) \\ \text{over the absolutely continuos arcs } x(\cdot), \\ \text{the measurable control functions } (u_0(\cdot), \ldots, u_n(\cdot)) \text{ and} \\ \text{the measurable vector functions } (\lambda_0(\cdot), \ldots, \lambda_n(\cdot)) \text{ s.t.} \\ \dot{x}(t) = \sum_{j=0}^{n} \lambda_j(t) f(t, x(t), u_j(t)) \quad \text{a.e. } t \in [0,1] \\ (u_0(t), \ldots, u_n(t)) \in U^{n+1}(t), \quad \text{a.e. } t \in [0,1] \\ (\lambda_0(t), \ldots, \lambda_n(t)) \in \Lambda, \quad \text{a.e. } t \in [0,1] \\ (x(0), x(1)) \in C_0 \times C_1 \end{cases}
$$

where

$$
\Lambda := \{(\lambda_0, \ldots, \lambda_n) \in \mathbb{R}^{n+1} : \lambda_i \geq 0 \; \forall i, \; \sum_{i=0}^{n} \lambda_i = 1\}.
$$

In this case, the triple $\left(x(\cdot), \{\lambda_i(\cdot), u_i(\cdot)\}_{k=0}^{n}\right)$ is a relaxed process.

Equivalence of the formulations

$i)$ Warga's approach $\Longrightarrow$ Convexification approach

If we choose a probability measure $t \to \mu(t)$ with support

$$\text{supp}\{\mu\}(t) = \{u_0, \ldots, u_n\}(t) \quad \text{a.e. } t \in [0,1]$$

and we define

$$\mu\{u_j\}(t) := \lambda_j(t) \quad \text{a.e. } t \in [0,1],$$

then it follows that

$$\int_{U(t)} f(t, x(t), r)\mu(t)(dr) = \sum_{j=0}^{n} \lambda_j(t) f(t, x(t), u_j(t)).$$

$ii)$ Convexification Approach $\Longrightarrow$ Warga's Approach
Consider the set

$$\text{co}\,\{f(t,x,U(t))\}$$

and define

$$a(t) := \int\limits_{U(t)} f(t,x,r)\mu(t)(dr).$$

If we suppose that

$$a(t) \notin \text{co}\,\{f(t,x,U(t))\}\,,$$

then there exists an $r > 0$ s.t.

$$B_r(a(t)) \cap \text{co}\,\{f(t,x,U(t))\} = \emptyset$$

and from the Separation Theorem:

there exists $p(t) \neq 0$ s.t.

$$p(t) \cdot d(t) \leq p(t) \cdot y(t)$$

for any $\quad d(t) \in B_r(a(t)),\ y(t) \in \text{co}\,\{f(t,x,U(t))\}\,.$

Superior extremum on the l.h.s. plus $y(t) = f(t,x,u(t))$ give:

$$p(t) \cdot a(t) + r|p(t)| \leq p(t) \cdot f(t,x,u(t)).$$

Then we integrate both sides

$$p(t) \cdot a(t) + r|p(t)| \leq \int_{U(t)} p(t) \cdot f(t,x,r)\mu(t)(dr) =$$

$$= p(t) \cdot \int_{U(t)} f(t,x,r)\mu(t)(dr) = p(t) \cdot a(t),$$

CONTRADICTION!!!

Relaxation Theorem

Define the original Reachable set

$\mathcal{R} := \{x(1) : \dot{x}(t) = f(t, x(t), u(t)), u(t) \in U(t)\text{a.e.}t \in [0, 1]\}$,

and the relaxed Reachable set

$$\mathcal{S} := \{x(1) : \dot{x}(t) = \int_{U(t)} f(t, x(t), r)\mu(t)(dr),$$

$$\mu(t) \in \text{r.p.m.}(U(t)) \text{ a.e. } t \in [0, 1]\}.$$

If we assume hypotheses $(H1)$ and $(H2)$, it follows that

$$\bar{\mathcal{R}} = \mathcal{S}.$$

An Example

Consider the problem:

$$(E)\begin{cases} \text{minimize}\, J(x) := \int_0^1 |x(t)|dt \\ \dot{x}(t) = u(t) \quad \text{a.e. } t \in [0,1] \\ u(t) \in \{-1, +1\} \quad \text{a.e. } t \in [0,1] \\ x(0) = 0 \end{cases}.$$

Consider the sequence of arcs $x_i(\cdot)$ related to the sequence of control function

$$u_i(s) = \begin{cases} +1 & \text{for} \quad s \in A_i \cap [0,1] \\ -1 & \text{for} \quad s \notin A_i \cap [0,1] \end{cases},$$

where

$$A_i = \bigcup_{j=0}^{\infty} \left[\frac{2j}{2i}, \frac{(2j+1)}{2i}\right].$$

It turns out that

$$J(x_i) = \int_0^1 dt \left| \int_0^t u_i(s) ds \right| \leq \frac{1}{2i},$$

which means that

$$\inf_{(E)} J(x) = 0.$$

If $(\bar{x}(\cdot), \bar{u}(\cdot))$ is a minimizer s.t. $J(\bar{x}) = \int_0^1 |\bar{x}(t)| dt = 0$,

$$\Longrightarrow$$

$\bar{x}(t) = 0$ and $\dot{\bar{x}}(t) = \bar{u}(t) = 0 \quad$ a.e. $t \in [0, 1]$.
CONTRADICTION!!!

Question

In general we have:

$$\inf_{(P)} g(x(1)) \geq \inf_{(P_{rel})} g(x(1))$$

When does the equivalence occur?

$$\inf_{(P)} g(x(1)) = \inf_{(P_{rel})} g(x(1))$$

Answer: When the minimizer $\bar{x}(\cdot)$ is not a boundary point of the constraint, which means

$$\text{or} \quad \bar{x}(0) + \varepsilon B \subset C_0, \quad \text{either} \quad \bar{x}(1) + \varepsilon B \subset C_1.$$