Hamilton Jacobi equations

HJ eq are nonlinear 1st order eqs which have been first introduced in classical mechanics but find applications in many other fields of maths
Our interest in these eqs lies mainey in the connection with calculus of variation and optimal control theory.
we consider the following HJ eq

$$\partial_t u(t,x) + H(t,x,D_x u(t,x)) = 0 \quad \text{on } \Omega \subset \mathbb{R}_+ \times \mathbb{R}^n \text{ open}$$
$$u(0,x) = u_0(x) \quad \text{on } \mathbb{R}^n \qquad (1)$$

HJ eq
Bellman's eq
Dynamic Programming equation

H is called the Hamiltonian usually is taken smooth
u_0 is the initial datum

METHOD OF CHARACTERISTICS

Using this method we will show that classical solution to this eq are in general not definite for infinite times
Suppose $H \in C^2$ and $u_0 \in C^2$
Given $z \in \mathbb{R}^n$ we define characteristic curve associated to u starting at the point z the curve $t \mapsto (t, X(t,z))$ such that

$$\dot X(t) = H_p(t, X(t), D_x u(t, X(t)) \quad X(0) = z$$

set
$$U(t,z) = u(t, X(t,z))$$
$$P(t,z) = D_x u(t, X(t,z))$$

Then
$$\dot U(t) = \partial_t u(t, X(t,z)) + D_x u(t, X(t,z))\, \dot X(t,z)$$
$$= -H(t, X(t,z), P(t,z)) + P(t,z) \cdot H_p(t, X(t,z), P(t,z))$$
$$\dot P(t) = D_x \partial_t u(t, X(t,z)) + D_x^2 u(t, X(t,z))\, \dot X(t, X(t,z))$$

now
$$0 = D_x(\partial_t u(t, X(t,z)) + H(t, X(t,z), D_x u(t, X(t,z)))) =$$
$$= D_x \partial_t u(t, X(t,z)) + H_x(t, X(t,z), D_x u(t, X(t,z)) +$$
$$+ D_x^2 u(t, X(t,z)) H_p(t, X(t,z), D_x u(t, X(t,z)))$$

Then $\dot P(t) = -H_x(t, X(t,z), P(t,z))$

Thus X and P solve

$$\begin{cases} \dot X(t) = H_p(t, X(t,z), P(t,z)) \\ \dot P(t) = -H_x(t, X(t,z), P(t,z)) \end{cases} \quad (2) \quad \begin{cases} X(0,z) = z \\ P(0,z) = D_x u_0(z) \end{cases}$$

and
$$\dot U(t) = -H(t, X(t,z), P(t,z)) + P(t,z) \cdot H_p(t, X(t,z), P(t,z)) \quad (3)$$
$$U(0,z) = u_0(z)$$

Therefore X, P, U are uniquely determined by the initial value u_0 and H.

Moreover one can obtain a solution to (1) by solving the char syst (2) and (3) provided the map $z \mapsto X(t,z)$ is invertible

THEOREM

For any $z \in \mathbb{R}^n$ let $X(t,z)$, $P(t,z)$ denote the solution of (2) and let $U(t,z)$ be defined by (3)

Suppose $\exists\, T^* > 0$ st

(I) The maximal interval of existence of the solution to (2) contains $[0,T^*)$ $\forall z \in \mathbb{R}^n$

(II) the map $z \mapsto X(t,z)$ is invertible with C^1 inverse $x \mapsto Z(t,x)$ $\forall t \in [0,T^*)$

Then there exists a unique $u \in C^2([0,T^*) \times \mathbb{R}^n)$ of (1) given by $u(t,x) = U(t, Z(t,x))$ $(t,x) \in [0,T^*) \times \mathbb{R}^n$

PROOF

For any $T < T^*$ $X(t,z)$ and $P(t,z)$ are well defined for (II) $z \mapsto X(t,z)$ is invertible with C^1 inverse $x \mapsto Z(t,x)$ $\forall t \in [0,T]$

Thus
$$X(t, Z(t,x)) = x \quad \forall (t,x) \in [0,T] \times \mathbb{R}^n$$
$$Z(t, X(t,z)) = z \quad \forall (t,z) \in [0,T] \times \mathbb{R}^n$$

set $u(t,x) = U(t, Z(t,x))$ $\forall (t,x) \in [0,T] \times \mathbb{R}^n$

Let us prove that $u(t,x)$ is a solution of (1)

$X(0,z) = z$ and $U(0,z) = u_0(z)$ $\Rightarrow$ $Z(0,x) = x$ and $u(0,x) = u_0(x)$

From the definition $u(t,x)$ is at least C^1

$$D_x u(t,x) = D_x(U(t, Z(t,x))) =$$
$$= U_z(t, Z(t,x)) \cdot Z_x(t,x)$$
$$\partial_t u(t,x) = U_t(t, Z(t,x)) + U_z(t, Z(t,x)) \cdot Z_t(t,x)$$

now
$$\dot U_z(t,z) = \frac{d}{dz}\Big(-H(t, X(t,z), P(t,z)) + P(t,z) \cdot H_p(t, X(t,z), P(t,z))\Big)$$
$$= -H_x X_z - \cancel{H_p P_z} + \cancel{P_z H_p} + p H_{px} X_z + p H_{pp} P_z$$
$$= \dot p \cdot X_z + p \cdot \underbrace{(H_{px} X_z + H_{pp} P_z)}_{} = \dot X_z$$
$$= (p \cdot X_z)^{\bullet} \qquad \text{from } U_z(0,z) = D_x u_0(z) = P(0,z) \cdot \overset{\text{Id}}{\overset{\|}{X_z(0,z)}}$$
$$\Rightarrow \quad U_z(t,z) = P(t,z) \cdot X_z(t,z)$$

$$D_x u(t,x) = U_z(t, Z(t,x))\cdot Z_x(t,x) = P(t,Z(t,x))\cdot \cancel{X_z(t,Z(t,x))\cdot Z_x(t,x)}$$
$$= P(t, Z(t,x))$$

$$\partial_t u(t,x) = U_t(t,Z(t,x)) + U_z(t, Z(t,x))\cdot Z_t(t,x)$$
$$= -H(t,x,D_x u(t,x)) + P(t,Z(t,x))\cdot H_p(t,x,P(t,Z(t,x))) +$$
$$+ P(t,Z(t,x))\, X_z(t,Z(t,x))\cdot Z_t(t,x)$$
$$= -H(t,x,D_x u(t,x))$$

indeed
$$\dot X(t,Z(t,x)) + X_z(t,Z(t,x))\cdot Z_t(t,x) =$$
$$= \frac{d}{dt}\left[X(t,Z(t,x))\right] = \frac{d}{\partial t}[x] = 0$$

Uniqueness follows from the fact that any other solution v can be determined by (2) with the same u_0 and U

$v(t,X(t,z)) = U(t,z) \quad \forall (t,z)$

$v(t,x) = v(t, X(t,Z(t,x))) = U(t,Z(t,x)) = u(t,x) \Rightarrow v = u$

T^* is the critical time after which characteristics start to cross $\Rightarrow$ discontinuities in the gradient occurs.

① Look for solutions which are defined only a.e. $\Rightarrow$ no uniqueness

EX
$$\begin{cases} \partial_t u + (D_x u)^2 = 0 & \mathbb{R}_+ \times \mathbb{R} \text{ ae} \\ u(0,x) = 0 & \mathbb{R} \end{cases}$$

$u \equiv 0$ is a solution but also $\forall a > 0$
$$u_a(t,x) = \begin{cases} 0 & |x| \geq at \\ a|x| - a^2 t & |x| < at \end{cases}$$

They are Lip solutions.

② A new concept of solution is necessary $\longrightarrow$ VISCOSITY SOLUTIONS

The Idea arise from vanishing viscosity method (no details)

DEF A bdd uniformly continuous function u is called viscosity solutions of (1) provided $u = u_0(x)$ on $\{t=0\}\times\mathbb{R}^n$

(i) u is a SUBSOLUTION $\forall v \in C^\infty((0,\infty)\times\mathbb{R}^n)$
$u - v$ has a local maximum in (t_0, x_0)
$$\partial_t v(t_0,x_0) + H(t_0, x_0, D_x v(t_0,x_0)) \leq 0$$

(ii) u is a SUPERSOLUTION $\forall v \in C^\infty((0,\infty)\times\mathbb{R}^n)$
$u - v$ has a local minimum in (t_0, x_0)
$$\partial_t v(t_0,x_0) + H(t_0, x_0, D_x v(t_0,x_0)) \geq 0$$

CONSISTENCY

PRO A classical solution is also a viscosity solution

PROOF

$\forall v$ smooth $u-v$ has a local maximum at (t_0, x_0)

$$D_x u(t_0, x_0) = D_x v(t_0, x_0).$$
$$\partial_t u(t_0, x_0) = \partial_t v(t_0, x_0)$$

Any sufficientey smooth viscosity solution is a classical sol

LEMMA Assume $u: \mathbb{R}^n \to \mathbb{R}$ continuous and differentiabee in x_0
Then $\exists\, v \in C^1(\mathbb{R}^n)$ st $u(x_0) = v(x_0)$ and $u-v$ has a strict local maximum at x_0.

THEOREM

Let u be a viscosity solution of (1) suppose u is differentiabee at some (t_0, x_0) then $\partial_t u(t_0, x_0) + H(t_0, x_0, D_x u(t_0, x_0)) = 0$

PROOF

From LEMMA $\exists\, v \in C^1(\mathbb{R}^{n+1})$ st $u-v$ has a strict max in (t_0, x_0)

$v^\varepsilon = \eta_\varepsilon * v$ $\quad \eta_\varepsilon$ usual mollifier in (t,x)

$$\left\{\begin{array}{l} v^\varepsilon \to v \\ D_x v^\varepsilon \to D_x v \\ \partial_t v^\varepsilon \to \partial_t v \end{array}\right. \quad \text{uniformey at } (t_0, x_0)$$

$u - v^\varepsilon$ has a maximum at some $t^\varepsilon, v^\varepsilon$

$$\partial_t v^\varepsilon(t^\varepsilon, x^\varepsilon) + H(t^\varepsilon, x^\varepsilon, D_x v^\varepsilon(t^\varepsilon, x^\varepsilon)) \leq 0$$

$\varepsilon \to 0$

$$\partial_t v(t_0, x_0) + H(t_0, x_0, D_x v(t_0, x_0)) \leq 0$$

U is diff in (t_0, x_0) $\Rightarrow$ $\partial_t v(t_0, x_0) = \partial_t u(t_0, x_0)$

$$D_x v(t_0, x_0) = D_x u(t_0, x_0)$$

$$\partial_t u(t_0, x_0) + H(t_0, x_0, D_x u(t_0, x_0)) \leq 0$$

Appeying the Lemma to $-u$ we obtain the opposite inequality

THEOREM

Let H be Lipschitz continuous in (t, x, p) then there exists at most one viscosity solution

THE VALUE FUNCTION

From the calculus of variation we consider a minimization pb with one free end point.

For a given $T>0$ we consider $\Omega = (0,T) \times \mathbb{R}^n$

Running cost $L : (0,T) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

Initial cost $u_0 : \mathbb{R}^n \to \mathbb{R}$

HYPOTHESES ON L and u_0

(u_0) u_0 bdd Lipschitz (first C^1)

(L1) $L \in C^2([0,T] \times \mathbb{R}^n \times \mathbb{R}^n)$

(L2) $\exists\, a, b, c > 0$ (I) $L(t,x,v) \geq -c$ (II) $L_x(t,x,0) \leq c$ (III) $|L_{vx}(t,x,v)| \leq a + b|v$

(L3) $L_{vv}(t,x,v)$ positive definite $\forall\, t, x, v$

we consider the following class of admissible arcs

$$A_{t,x} = \{ \xi \in AC([0,t], \mathbb{R}^n) \mid \xi(t) = x \}$$

$$J_t(\xi) = u_0(\xi(0)) + \int_0^t L(s, \xi(s), \dot{\xi}(s))\, ds$$

$$u(t,x) = \min_{\xi \in A_{t,x}} J_t(\xi)$$

PROPOSITION PROPERTIES OF MINIMIZERS

Taken a minimizer ξ for (t,x)

(I) $s \mapsto L_v(s, \xi(s), \dot{\xi}(s))$ is absolutely continuous

(II) ξ is a classical solution to Euler-Laguange eq

$$\frac{d}{ds} L_v(s, \xi(s), \dot{\xi}(s)) = L_x(s, \xi(s), \dot{\xi}(s)) \quad (4)$$

and to du Bois-Reymond eq

$$\frac{d}{ds}\left[L(s, \xi(s), \dot{\xi}(s)) - \langle \dot{\xi}(s), L_v(s, \xi(s), \dot{\xi}(s)) \rangle \right] = L_t(s, \xi(s), \dot{\xi}(s))$$

(III) $\xi \in C^2([0,t], \mathbb{R}^n)$

PROPOSITION

For any $r > 0$ $\exists\, M(r) > 0$ st if $(t,x) \in [0,T] \times B_r$ and ξ is a minimizer for (t,x) then

$$\sup_{s \in [0,t]} |\xi(s)| \leq M(r)$$

finite speed of Propagation

DEF Any arc $\xi \in A_{t,x}$ which solves (4) and such that
$L_v(0, \xi(0), \dot\xi(0)) = D_x u_0(\xi(0))$ is called extremal

(4) can be restated as a 1st order system of 2n eqs

Hamiltonian $\quad H(t,x,p) = \max\limits_{v \in \mathbb{R}^n} [\langle p,v\rangle - L(t,x,v)]$ Legendre transform

PROPERTIES OF THE LEGENDRE TRANSFORM

$L \in C([0,T] \times \mathbb{R}^n \times \mathbb{R}^n)$

L convex in v and $\quad \liminf\limits_{|v| \to +\infty} \inf\limits_{t,x} \dfrac{L(t,x,v)}{|v|} = +\infty \quad$ (★)

$$H(t,x,p) = \sup_{v \in \mathbb{R}^n} \langle p,v\rangle - L(t,x,v)$$

THEOREM

(I) $\forall (t,x,p)$ The sup is a max. For every bdd set $C \subseteq \mathbb{R}^n$
$\exists R > 0$ s.t. $\forall v$ associated to $(t,x,p) \in [0,T] \times \mathbb{R}^n \times C \Rightarrow |v| \le R$

(II) H is continuous convex and $\lim\limits_{|p| \to +\infty} \inf\limits_{t,x} \dfrac{H(t,x,p)}{|p|} = +\infty$

(III) if L is strictly convex in v then $\dfrac{\partial H}{\partial p}(t,x,p)$ exists and is continuous

(IV) $L(t,x,v) = \sup\limits_{p \in \mathbb{R}^n} \langle p,v\rangle - H(t,x,p)$

THEOREM Let $L \in C^k$ $k \ge 2$ and (★) and $\dfrac{\partial^2 L}{\partial v^2}(t,x,v)$ pos def

Then $H \in C^k$ denote with $v(t,x,p)$ the unique max

$$\frac{\partial H}{\partial p}(t,x,p) = v(t,x,p)$$

$$\frac{\partial H}{\partial x}(t,x,p) = -\frac{\partial L}{\partial x}(t,x,v(t,x,p))$$

$$\frac{\partial^2 H}{\partial p^2}(t,x,p) = \left[\frac{\partial^2 L}{\partial v^2}(t,x,v(t,x,p))\right]^{-1}$$

$$v = \frac{\partial H}{\partial p}(t,x,p) \iff p = \frac{\partial L}{\partial v}(t,x,v)$$

THEOREM

Let $\xi \in C^2([0,t]; \mathbb{R}^n)$ be extremal for our pb and set

$\eta(s) = L_v(s, \xi(s), \dot{\xi}(s)) \quad s \in [0,t]$

then $\eta(0) = D_x u_0(\xi(0))$ and ξ, η statisfy

$$\begin{cases} \dot{\xi}(s) = H_p(s, \xi(s), \eta(s)) \\ \dot{\eta}(s) = -H_x(s, \xi(s), \eta(s)) \end{cases} \quad (5)$$

Conversely suppose that $(\xi, \eta) \in C^2([0,t], \mathbb{R}^n)$ solve (5) with $\xi(t) = x \quad \eta(0) = D_x u_0(\xi(0))$ Then ξ is extremal

PROOF

Since L_v and H_p are reciprocal inverses

$\eta(s) = L_v(s, \xi(s), \dot{\xi}(s)) \Rightarrow \dot{\xi}(s) = H_p(s, \xi(s), \eta(s)) \quad s \in [0,t]$

ξ extremal $\Rightarrow$ EU-LAG + normality condition

$\Rightarrow \dot{\eta}(s) = L_x(s, \xi(s), \dot{\xi}(s)) \quad \eta(0) = Du_0(\xi(0))$

Recalling $-H_x(t,x,L_v(t,x,v)) = L_x(t,x,v)$ we prove the 1st PART

The converse is obtained by similar arguments.

DEF η is dual arc or costate (terminology of control theory)

ACTION ANGLE MECH

DYNAMIC PROGRAMMING PRINCIPLE

Fix (t,x) then for any $t' \in [0,t]$

$$u(t,x) = \min \left\{ u(t', \xi(t')) + \int_{t'}^{t} L(s, \xi(s), \dot{\xi}(s))\, ds \;\middle|\; \begin{array}{l} \xi(t) = x \\ \xi \in AC([t',t], \mathbb{R}^n) \end{array} \right\}$$

PROOF

take $\xi \in A_{t,x}$

fixed $t' \in [0,t]$ let $\tilde{\xi}$ any curve in $AC[0,t']$ st $\tilde{\xi}(t') = \xi(t')$

set $\hat{\xi}(s) = \begin{cases} \tilde{\xi}(s) & s \in [0,t'] \\ \xi(s) & s \in [t',t] \end{cases} \qquad \hat{\xi}(s) \in AC([0,t], \mathbb{R}^n)$

$$u(t,x) \leq \left[u_0(\tilde{\xi}(0)) + \int_0^{t'} L(s, \tilde{\xi}(s), \dot{\tilde{\xi}}(s))\, ds \right] + \int_{t'}^{t} L(s, \xi(s), \dot{\xi}(s))\, ds$$

and taking the min over all $A_{t', \xi(t')}$ we obtain

$$u(t,x) \leq u(t', \xi(t')) + \int_{t'}^{t} L(s, \xi(s), \dot{\xi}(s))\, ds$$

Now if the above ineq is an equality for all $t' \Rightarrow$ for $t' = 0$ ξ is a minimizer for $u(t,x)$

Conversely if ξ is a minimizer for $u(t,x)$

$u_0(\xi(0)) + \int_0^t L(s,\xi(s),\dot\xi(s))\,ds = u(t,x) \leq u(t',\xi(t')) + \int_{t'}^t L(s,\xi(s),\dot\xi(s))$

$\Rightarrow \quad u_0(\xi(0)) + \int_0^{t'} L(s,\xi(s),\dot\xi(s))\,ds \leq u(t',\xi(t'))$

$\geq$ always holds by def $\Rightarrow$ $=$ holds for every t'

For the opposite inequality

given $\varepsilon > 0$ let $\xi \in A_{t,x}$ st

$u(t,x) + \varepsilon \geq u_0(\xi(0)) + \int_0^t L(s,\xi(s),\dot\xi(s))\,ds$

Then

$$u(t,x) \geq u_0(\xi(0)) + \int_0^{t'} L(s,\xi(s),\dot\xi(s))\,ds + \int_{t'}^t L(s,\xi(s),\dot\xi(s))\,ds - \varepsilon$$

$$\geq u(t',\xi(t')) + \int_{t'}^t L(s,\xi(s),\dot\xi(s))\,ds - \varepsilon \qquad \varepsilon \text{ is arbitrary}$$

$$u(t,x) \geq u(t',\xi(t')) + \int_{t'}^t L(s,\xi(s),\dot\xi(s))\,ds$$

SEMICONCAVE FUNCTIONS

DEF A function $f: \Omega \to \mathbb{R}$ $\Omega \subseteq \mathbb{R}^n$ is said to be semiconcave if $\exists C > 0$ st for any $x, z \in \Omega$ st $[x-z, x+z] \subset \Omega$ $\qquad u \in SC(\Omega)$

$$u(x+z) + u(x-z) - 2u(x) \leq c|z|^2 \quad (6)$$

PROPOSITION Let $u \in SC(\Omega)$ $C \geq 0$ Ω open convex then

$\tilde u : x \mapsto u(x) - \frac{c}{2}|x|^2$ is concave

ie $\forall x, y \in \Omega$ $[x,y] \subset \Omega$ $\lambda \in [0,1]$

$$\tilde u(\lambda x + (1-\lambda)y) \geq \lambda \tilde u(x) + (1-\lambda)\tilde u(y)$$

PROOF

LEMMA $u: A \to \mathbb{R}$ A open convex u convex iff

$u(x+h) + u(x-h) - 2u(x) \geq 0$ for all x, h st $x \pm h \in A$

Using the identity $|x+h|^2 + |x-h|^2 - 2|x|^2 = 2|h|^2$

(6) is equivalent to $\tilde u(x+h) + \tilde u(x-h) - 2\tilde u(x) \leq 0$

DEF $u: \Omega \to \mathbb{R}$ $\Omega \subseteq \mathbb{R}^n$

$$D^- u(x) = \left\{ p \in \mathbb{R}^n \;\middle|\; \liminf_{y \to x} \frac{u(y) - u(x) - \langle p, y-x \rangle}{|y-x|} \geq 0 \right\}$$

$$D^+ u(x) = \left\{ p \in \mathbb{R}^n \;\middle|\; \limsup_{y \to x} \frac{u(y) - u(x) - \langle p, y-x \rangle}{|y-x|} \leq 0 \right\}$$

$D^- u$ $D^+ u$ are closed convex

They are both non empty iff u is diff in x

$$D^- u(x) = D^+ u(x) = Du(x)$$

u locally lip $p \in \mathbb{R}^n$ is called reachable gradient of u at $x \in \Omega$ if there exists a seq $\{x_n\} \subset \Omega \setminus \{x\}$ st u is diff at x_k $\forall k \in \mathbb{N}$ and $\lim_{k \to +\infty} x_k = x$ $\lim_{k \to +\infty} Du(x_k) = p$

D^*u is the set of all reachable gradients

THEOREM $u : \Omega \to \mathbb{R}$ $u \in SC(\Omega)$ Then

(I) Alexandroff theorem u is twice differentiable $\mathcal{H}^n$ a e

(II) $Du \in BV_{loc}(\Omega)$

(III) let $x \in \Omega$ then $D^+u(x) = co\, D^*u(x)$ $\quad co\, A = \{ B \mid B \supset A \text{ A convex}\}$

(IV) $T(x) = -D^+\tilde{u}$ is a maximal monotone function

$\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0 \quad \forall x_i \in \Omega \; y_i \in T(x_i)$

$V \supset T$ V monotone $\Rightarrow V = T$

THEOREM 1 L1-L3 Hold $u_0 \in C(\mathbb{R}^n)$

Then for any $t \in (0,T]$ $u(t,\cdot) \in SC_{loc}(\mathbb{R}^n)$

PROOF

for fixed $t, r > 0$ let $x, z \in \mathbb{R}^n$ st $x \pm z \in B_r$ ξ minimizing curve for (t,x)

$\xi_\pm(s) = \xi(s) \pm \frac{s}{t} z$ $\quad L \in C^2$ is loc semiconcave

$$u(t,x+z) + u(t,x-z) - 2u(t,x) \leq \cancel{u_0(\xi_+(0))} + \int_0^t L(s, \xi_+, \dot{\xi}_+)ds$$
$$+ \cancel{u_0(\xi_-(0))} + \int_0^t L(s, \xi_-, \dot{\xi}_-)ds - \cancel{2u_0(\xi(0))} - 2\int_0^t L(s,\xi,\dot{\xi})ds$$
$$\leq \int_0^t \Big[L\Big(s, \xi(s) + \frac{s}{t}z, \dot{\xi}(s) + \frac{z}{t}\Big) + L\Big(s, \xi(s) - \frac{s}{t}z, \dot{\xi}(s) - \frac{z}{t}\Big) -2L\big(s, \xi(s), \dot{\xi}(s)\big)\Big] ds$$
$$\leq \int_0^t C \, \frac{s^2+1}{t^2} |z|^2 ds = C\left(\frac{t}{3} + \frac{1}{t}\right)|z|^2$$

THEOREM 2 L1-L3 hold $u_0 \in SC_{loc}(\mathbb{R}^n) \Rightarrow u \in SC_{loc}([0,T] \times \mathbb{R}^n)$

COROLLARY L1-L3 $\Rightarrow u \in SL_{loc}((0,T] \times \mathbb{R}^n)$

PROOF

Fix $t' \in (0,T]$ from THEO1 and dynamic programming $u|_{[t',T] \times \mathbb{R}^n}$ can be rep as the value function of a pb with a semiconcave initial cost $\Rightarrow$ is semiconcave by THEO2

DEF Equivalent def of viscosity solution $u(t,x)$ is a v.s if

$P_t + H(t,x,P_x) \le 0 \quad \forall (P_t,P_x) \in D^+u(t,x)$

$P_t + H(t,x,P_x) \geq 0 \quad \forall (P_t,P_x) \in D^-u(t,x)$

PROPOSITION

$P \in D^+u(x) \Leftrightarrow p = D\phi(x)$ for some $\phi \in C^1$ st $u-\phi$ have a local max at x
$D^-u(x)$ min

THEOREM

The value function is a viscosity solution of (1)

PROOF

take $(t,x) \in [0,T]\times\mathbb{R}^n$ for any $P_t, P_x \in D^+u(t,x)$ $\forall v \in \mathbb{R}^n$ we have

$$\limsup_{h\to 0^+} \frac{u(t-h,x-hv)-u(t,x)+h(P_t+\langle v,P_x\rangle)}{h\sqrt{1+|v|^2}} \le 0$$

$$\Leftrightarrow \limsup_{h\to 0^+} \frac{u(t-h,x-hv)-u(t,x)}{h} \leq -P_t - \langle v,P_x\rangle$$

$\zeta(\sigma) = x+(\sigma-t)v \quad \sigma\le t$ from dynamic programming

$$u(t,x) \le u(t-h,\zeta(t-h)) + \int_{t-h}^t L(\sigma,\zeta(\sigma),\dot\zeta(\sigma))\,d\sigma$$
$$= u(t-h,x-hv) + \int_{t-h}^t L(\sigma, x+(\sigma-t)v, v)\,d\sigma$$

$$\limsup_{h\to 0^+} \frac{u(t-h,x-hv)-u(t,x)}{h} \geq \lim_{h\to 0^+} \frac{1}{h}\int_{t-h}^t L(\sigma,x+(\sigma-t)v,v)\,d\sigma$$
$$= -L(t,x,v)$$

$\Rightarrow P_t + v\cdot P_x - L(t,x,v) \le 0 \quad \forall v$

$\Rightarrow P_t + H(t,x,P_x) \le 0$ is a sub solution

now $\forall (P_t,P_x) \in D^-u(t,x)$ let $\xi \in C^1([0,t],\mathbb{R}^n)$ be a minimizer for $u(t,x)$

$w = \dot\xi(t)$ from the lipschitz continuity of u (min semiconcave)

$$\liminf_{h\to 0^+} \frac{u(t-h,\xi(t-h))-u(t,x)}{h} = \liminf_{h\to 0^+} \frac{u(t-h,x-hw)-u(t,x)}{h}$$
$$\geq -P_t - wP_x$$

On the other hand ξ optimal

$$u(t,x) = u(t-h,\xi(t-h)) + \int_{t-h}^t L(s,\xi(s),\dot\xi(s))\,ds \quad \forall 0<h<t$$

$$\Rightarrow \lim_{h\to 0^+} \frac{u(t-h,\xi(t-h))-u(t,x)}{h} = -L(t,x,w)$$

$$P_t + wP_x - L(t,x,w) \geq 0 \Rightarrow P_t + H(t,x,P_x) \geq 0$$

Moreover u stsfs the initial condition

THEOREM

Let $(t,x) \in [0,T] \times \mathbb{R}^n$ given a minimizing curve ξ for (t,x) Then u is differentiable at $(s, \xi(s))$ $\forall s \in (0,t)$

THEOREM

η defined as in (5) then $\eta(t) \in D^+_x u(t,x)$

$$\eta(s) = D_x u(s, \xi(s)) \quad \forall s \in (0,t)$$

THEOREM

$\forall (t,x) \in [0,T] \times \mathbb{R}^n$ the map that associates with any $(p_t, p_x) \in D^* u(t,x)$ the arc ξ obtained solving the system (5)

$$\begin{cases} \xi(t) = x \\ \psi(t) = p_x \end{cases}$$

provides a one to one correspondence between $D^* u(t,x)$ and the set of minimizers for $u(t,x)$

CORO $u(t,x)$ is differentiable iff the minimizer is unique