# Reduced-order modeling for control design

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# Active vibration control of slender structures

$$\partial^2_{xx}(EI\partial^2_{xx}w) = -\mu\partial^2_{tt}w + q(x)$$

Figure 1. An example of direct-active system: F-18 Fighter instrumented fin with piezoelectric actuators for buffeting control. From Nitzsche et al. (2001).

# Active noise cancellation

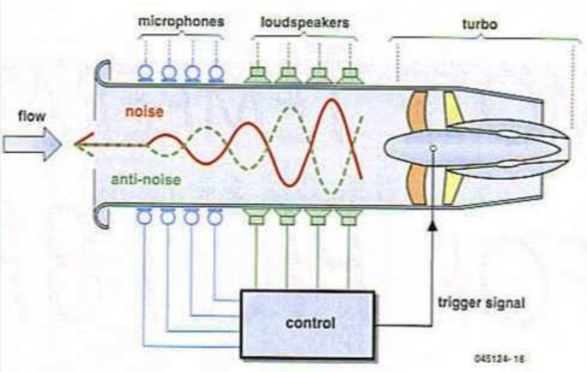
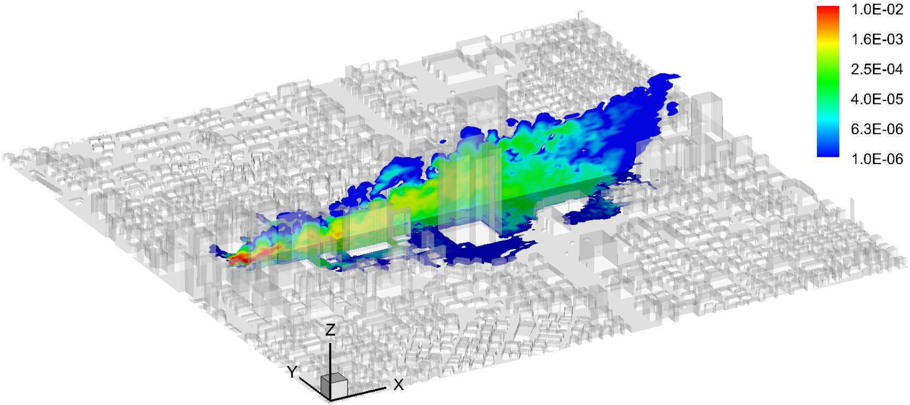



*Figure 7.*
*Active noise control at source: research at the German Aerospace Centre.*

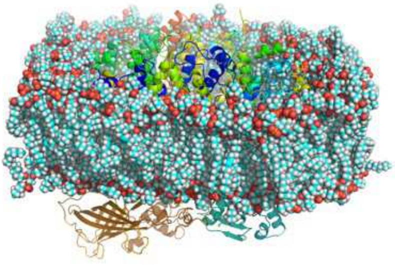
# Pollution propagation

$$\partial_t c_i + \nabla \cdot (\mathbf{u}\, c_i) - \nabla \cdot (\mathbf{K}\, \nabla c_i) = R_i(c_1, \ldots, c_d) + S_i$$



# Molecular dynamics

$$\partial_{tt}^2 \mathbf{x} = -\nabla\phi(\mathbf{x})$$



# Some references


- A. Antoulas. Approximation of Large-Scale Dynamical Systems, SIAM, 2005 .
- S. Volkwein. Proper Orthogonal Decomposition: Theory and Reduced-Order Modelling, Lecture Notes, available at: `http://w3.math.uni-konstanz.de/numerik/personen/volkwein/`

# The need of model reduction

Reasons for ROM:

- Simulation.
- The computation of the feedback is intensive on resources.

- Real time control requires controllers of low complexity!

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**Our "Ultimate Goal"**

Is it possible to recover an optimal controller of lower complexity, computed from an approximate model that mimics the I-O behavior of the original system?

# Outline

# Overview

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# Reachability and Observability

Consider a system described in state-space form $\sum = \sum(\mathcal{A}, \mathcal{B}, \mathcal{C})$.

- Let $h_r(t) = e^{\mathcal{A}t}\mathcal{B}$ be the I-S response.
- Let $h_o(t) = \mathcal{C}e^{\mathcal{A}t}$ be the S-O response.

**Reachability gramian**

Given $\mathcal{P} = \int_0^\infty h_r(t) h_r^T(t)\,dt$ The smallest amount of energy to move the system from $0$ to $x$ is $\epsilon_r = x^T \mathcal{P}^{-1} x$.

**Observability gramian**

Given $\mathcal{Q} = \int_0^\infty h_o^T(t) h_o(t)\,dt$ The amount of observed energy from a free system with initial condition $x$ is $\epsilon_o = x^T \mathcal{Q} x$.

# Overview

# Balancing

Assuming that $S(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is exponentially stabilizable and detectable,

## Lyapunov equations and balanced realization

The reachability and controlability gramians are the unique, self-adjoint positive semidefinite solutions $\mathcal{P}$ and $\mathcal{Q}$ that satisfy the following Lyapunov equations

$$\mathcal{A}\mathcal{P} + \mathcal{P}\mathcal{A}^T + \mathcal{B}\mathcal{B}^T = 0,$$

$$\mathcal{A}^T\mathcal{Q} + \mathcal{Q}\mathcal{A} + \mathcal{C}^T\mathcal{C} = 0,$$

We look for a balancing transformation such that

$$\mathcal{P} = \mathcal{Q} = \Sigma = diag(\sigma_j)$$

.

# Solution method

We determine the Cholesky factors $R$ and $L$ of the Gramians, i.e.

$$\mathcal{P} = RR^T, \quad \mathcal{Q} = LL^T.$$

Further, we compute the singular value decomposition

$$L^T R = (U_1 \;\; U_2) \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

with orthonormal matrices $U = (U_1 \;\; U_2)$ and $V = (V_1 \;\; V_2)^T$ and diagonal matrices

$$\Sigma_1 = \mathrm{diag}(\sigma_1, \ldots, \sigma_r), \quad \Sigma_2 = \mathrm{diag}(\sigma_{r+1}, \ldots, \sigma_l)$$

with

$$\sigma_1 \geq \cdots \geq \sigma_r \gg \sigma_{r+1} \geq \cdots \geq \sigma_l > 0$$

and $l = \mathrm{rank}(L^T R)$.

# Truncation

The singular values of $L^T R$ provide a measure of the energy of the corresponding balanced state. By setting

$$W = LU_1\Sigma_1^{-\frac{1}{2}}, \quad S = RV_1\Sigma_1^{-\frac{1}{2}}$$

we can compute

$$\mathcal{A}^r = W^T\mathcal{A}S, \quad \mathcal{B}^r = S^T\mathcal{B}, \quad \mathcal{C}^r = \mathcal{C}W$$

## Reduced-order model

$$\begin{cases} \partial_t y^r = \mathcal{A}^r y^r + \mathcal{B}^r u, \\ y^r(0) = W^T x, \\ z^r = \mathcal{C}^r y^r \end{cases}$$

## Error bound

$$\|z^r - z\|_{L^2(0,\infty)} \leq 2(\sigma_{r+1} + \cdots + \sigma_l)\,\|u\|_{L^2(0,\infty)}.$$

# Balancing

## Balancing alternative

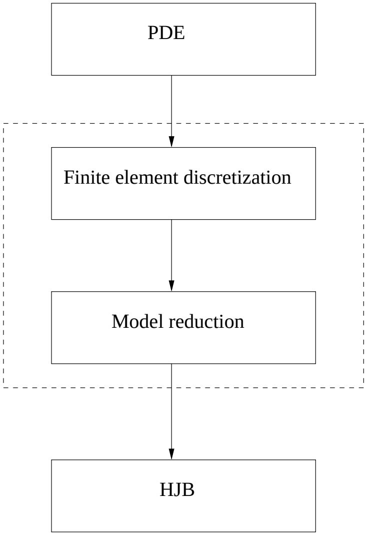
We compute the Cholesky factor $U$ of $P$ and the eigenvalue decomposition of $U^*\mathcal{Q}U$:

$$\mathcal{P} = UU^*, \qquad U^*\mathcal{Q}U = K\Sigma^2K^*$$

There exists then a balancing transformation given by $T = \Sigma^{1/2}K^*U^{-1}$ and $T^{-1} = UK\Sigma^{-1/2}$.

# Overview

# Design steps



# Numerical approximation

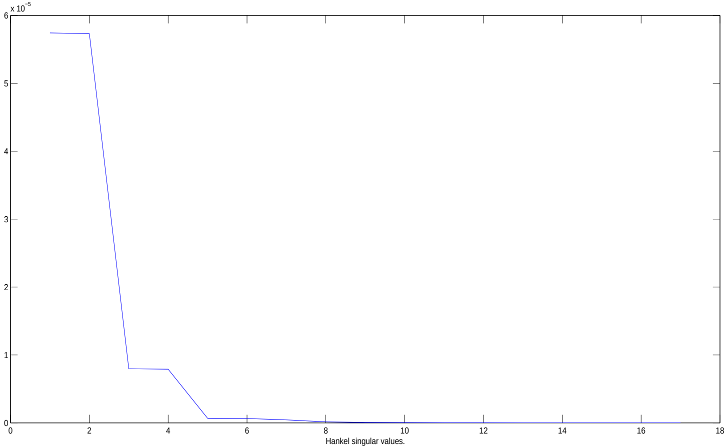
Numerical tests:

- Wave equation. Distributed control operator.
- Observation operator $Cx = \int_{\Omega_{obs}} x(:, t) d\xi$

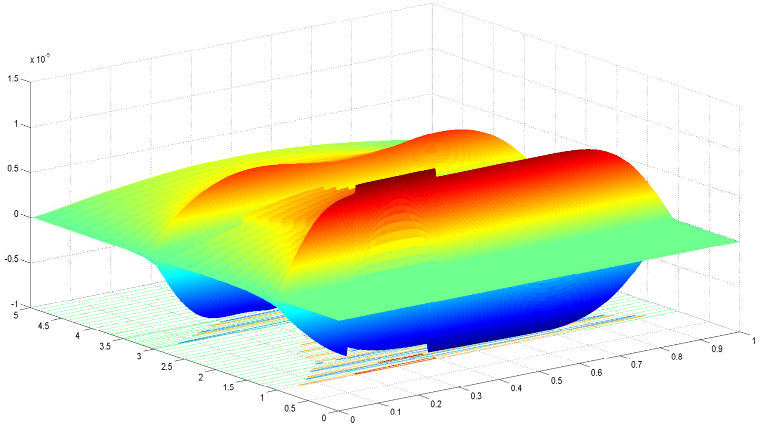
Procedures:

1. FEM-ROMBT-IO (step).
2. FEM-ROMBT-LQR.

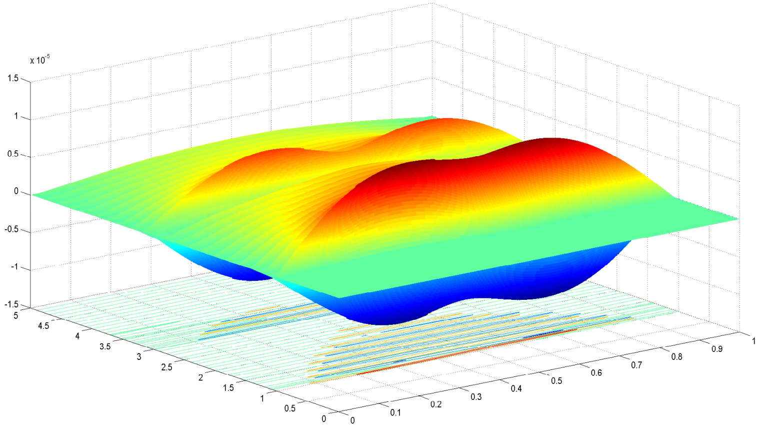
# Hankel singular values



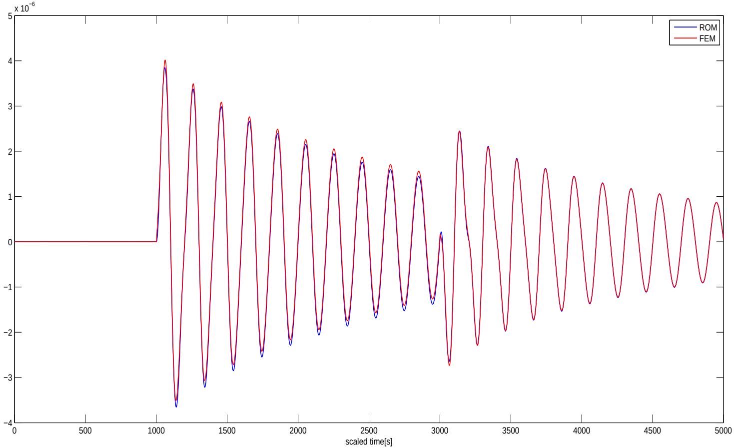
# FEM



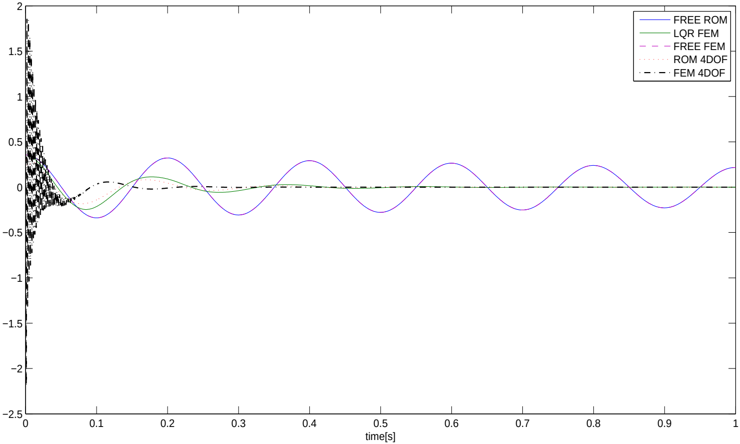
# Recovered ROM dynamics (4 states)



# FEM and ROM observations (4 states)



# FEM and ROM LQR and free outputs (4 states)



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